

EXISTENCE OF A MARTINGALE SOLUTION OF THE STOCHASTIC NAVIER-STOKES EQUATIONS IN UNBOUNDED 2D AND 3D-DOMAINS.

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ABSTRACT. Stochastic Navier-Stokes equations in 2D and 3D possibly unbounded domains driven by a multiplicative Gaussian noise are considered. The noise term depends on the unknown velocity and its spatial derivatives. The existence of a martingale solution is proved. The construction of the solution is based on the classical Faedo-Galerkin approximation, the compactness method and the Jakubowski version of the Skorokhod Theorem for nonmetric spaces. Moreover, some compactness and tightness criteria in nonmetric spaces are proved. Compactness results are based on a certain generalization of the classical Dubinsky Theorem.

Keywords: Stochastic Navier-Stokes equations, martingale solution, compactness method

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1. INTRODUCTION.

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open connected possibly unbounded subset with smooth boundary $\partial\mathcal{O}$, where $d = 2, 3$. We consider the stochastic Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(t) + G(t, u) dW(t), \quad t \in [0, T],$$

in \mathcal{O} , with the incompressibility condition

$$\operatorname{div} u = 0$$

and with the homogeneous boundary condition $u|_{\partial\mathcal{O}} = 0$. In this problem $u = u(t, x) = (u_1(t, x), \dots, u_d(t, x))$ and $p = p(t, x)$ represent the velocity and the pressure of the fluid. Furthermore, f stands for the deterministic external forces and $G(t, u) dW(t)$, where W is a cylindrical Wiener process, stands for the random forces.

The above problem can be written in an abstract form as the following initial value problem

$$\begin{cases} du(t) + \mathcal{A}u(t) dt + B(u(t)) dt = f(t) dt + G(u(t)) dW(t), & t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Here \mathcal{A} and B are appropriate maps corresponding to the Laplacian and the nonlinear term, respectively in the Navier-Stokes equations, see Section 2. We impose rather general assumptions (G), (G*) and (G**) on the noise $G(t, u)dW(t)$ formulated in (A.3) in Section 4. These assumptions cover the following special case

$$G(t, u)dW(t) := \sum_{i=1}^{\infty} [(b^{(i)}(x) \cdot \nabla)u(t, x) + c^{(i)}(x)u(t, x)] d\beta^{(i)}(t),$$

where $\{\beta^{(i)}\}_{i \in \mathbb{N}}$ are independent real standard Brownian motions, see Section 6.

We prove the existence of a martingale solution. The construction of a solution is based on the classical Faedo-Galerkin approximation, i.e.

$$\begin{cases} du_n(t) = -[P_n \mathcal{A}u_n(t) + B_n(u_n(t)) - P_n f(t)] dt + P_n G(u_n(t)) dW(t), \\ u_n(0) = P_n u_0 \end{cases}$$

given in Section 5. The crucial point is to prove suitable uniform *a priori* estimates on u_n . Analogously to [21], we prove that the following estimates hold

$$\sup_{n \geq 1} \mathbb{E} \left[\int_0^T \|u_n(s)\|_V^2 ds \right] < \infty.$$

and

$$\sup_{n \geq 1} \mathbb{E} \left(\sup_{0 \leq s \leq T} |u_n(s)|_H^p \right) < \infty.$$

for $p \in [2, 2 + \frac{\eta}{2-\eta})$, where $\eta \in (0, 2)$ is given parameter, see Section 4. Here, V and H denote the closures in the Sobolev space $H^1(\mathcal{O}, \mathbb{R}^d)$ and $L^2(\mathcal{O}, \mathbb{R}^d)$, respectively of the divergence-free vector fields of class \mathcal{C}^∞ with compact supports contained in \mathcal{O} . The solutions u_n to the Galerkin scheme generate a sequence of laws $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$ on appropriate functional spaces. To prove that this sequence of probability measures is weakly compact we need appropriate tightness criteria.

In Section 3 we prove certain deterministic compactness results, see Lemmas 3.1 and 3.3. If \mathcal{O} is unbounded, then the embedding $V \hookrightarrow H$ is not compact. However, using Lemma 2.5 from [20] (see Lemma C.1 in Appendix C), we can find a separable Hilbert space U such that

$$U \subset V \subset H$$

the embedding $\iota : U \hookrightarrow V$ being dense and compact. Then we have

$$U \xhookrightarrow{\iota} H \cong H' \xhookrightarrow{\iota'} U',$$

where H' and U' are the dual spaces of H and U , respectively, H' being identified with H and ι' is the dual operator to the embedding ι . Moreover, ι' is compact as well. Modifying the proof of the classical Dubinsky Theorem, [38, Theorem IV.4.1], we obtain a certain deterministic compactness criterion, see Lemma 3.1. Namely, we will show that a set \mathcal{K} is relatively compact in the intersection

$$\tilde{\mathcal{Z}} := \mathcal{C}([0, T]; U') \cap L_w^2(0, T; V) \cap L^2(0, T; H_{loc})$$

if the following two conditions hold

- $\sup_{u \in \mathcal{K}} \int_0^T \|u(s)\|_V^2 ds < \infty$, i.e. \mathcal{K} is bounded in $L^2(0, T; V)$,
- $\lim_{\delta \rightarrow 0} \sup_{u \in \mathcal{K}} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \delta}} |u(t) - u(s)|_{U'} = 0$.

Here $\mathcal{C}([0, T]; U')$ denotes the space of U' -valued continuous functions, $L_w^2(0, T; V)$ is the space $L^2(0, T; V)$ equipped with the weak topology and $L^2(0, T; H_{loc})$ is a Fréchet space defined in (3.6) in Section 3. Let us notice that the second condition implies the equicontinuity of the family \mathcal{K} of U' -valued functions. Thus the above two conditions are the same as in the Dubinsky Theorem. However, since the embedding $V \hookrightarrow H$ is not compact, then in comparison to the Dubinsky Theorem we have the space $L^2(0, T; H_{loc})$ instead of $L^2(0, T; H)$.

Next, using this version of the Dubinsky Theorem, we will prove another deterministic compactness criterion, see Lemma 3.3. Namely, we will show that a set \mathcal{K} is relatively compact in the intersection

$$\mathcal{Z} := \mathcal{C}([0, T]; U') \cap L_w^2(0, T; V) \cap L^2(0, T; H_{loc}) \cap \mathcal{C}([0, T]; H_w)$$

if the following three conditions hold

- (a) $\sup_{u \in \mathcal{K}} \sup_{s \in [0, T]} |u(s)|_H^2 < \infty$,
- (b) $\sup_{u \in \mathcal{K}} \int_0^T \|u(s)\|^2 ds < \infty$,
- (c) $\lim_{\delta \rightarrow 0} \sup_{u \in \mathcal{K}} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \delta}} |u(t) - u(s)|_{U'} = 0$.

These results were inspired by Lemma 2.7 due to Mikulevicius and Rozovskii [31], where the case of $\mathcal{O} := \mathbb{R}^d$, $d \geq 2$ is considered. In [31] the space $L^2(\mathbb{R}^d)$ is compactly embedded in the Fréchet space $H_{loc}^{-k_0}(\mathbb{R}^d)$ for sufficiently large k_0 . Then, the authors prove the deterministic compactness criterion in the intersection

$$\mathcal{C}([0, T]; H_{loc}^{-k_0}(\mathbb{R}^d)) \cap \mathcal{C}([0, T]; L_w^2(\mathbb{R}^d)) \cap L_w^2(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; L_{loc}^2(\mathbb{R}^d)).$$

The main difference with the approach of Mikulevicius and Rozovskii, is that instead of the Fréchet space $H_{loc}^{-k_0}(\mathbb{R}^d)$, we consider the space U' dual to the Hilbert space U constructed in a special way by Holly and Wiciak, see [20, Lemma 2.5]. This allows us to prove the above mentioned modification

of the Dubinsky Theorem. The space U will be also of crucial importance in further construction of a martingale solution.

Using Lemma 3.3 and the Aldous condition in the form given by Métivier [30], we obtain a new tightness criterion for the laws on the space \mathcal{Z} , see Corollary 3.9. Next, we prove that the set of laws $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$ is tight on \mathcal{Z} . The next step in our construction of a martingale solution differs from the approach of Mikulevicius and Rozovskii. We apply the method used by Da Prato and Zabczyk in [16, Chapter 8]. This method is based on the Skorokhod Theorem and the martingale representation Theorem. However, we will apply the Jakubowski's version of the Skorokhod Theorem for nonmetric spaces in the form given by Brzeźniak and Ondreját [11], [23]. In [16, Chapter 8] the authors impose the linear growth conditions on the nonlinear term and assume the compactness of the appropriate semigroup. The assumptions considered in [16, Chapter 8] do not cover the stochastic Navier-Stokes equations, however, we can use the ideas introduced there. This method seems to us more direct. In the case of 2D domains we prove moreover the existence and uniqueness of strong solutions.

Stochastic Partial Differential Equations (SPDEs) can be viewed as an intersection of the infinite dimensional Stochastic Analysis and Partial Differential Equations. The theory of SPDEs began in the early 70's with works of Bensoussan-Temam [4], Dawson and Salehi [18] and many others. Due to contributions of several authors such as Pardoux [33] Krylov-Rozovskii [25], Da Prato-Zabczyk [16] many aspects of this new theory are now well developed and understood. The study of stochastic NSEs initiated in [4] was continued by many, for instance Brzeźniak et al [7, 8], Flandoli and Gatarek [21], Hairer-Mattingly [19] and Mikulevicius-Rozovskii [31]. In the last paper the authors study the existence of a martingale solution of the stochastic Navier-Stokes equations for turbulent flows in \mathbb{R}^d , ($d \geq 2$) corresponding to the Kraichnan model of turbulence.

Stochastic Navier-Stokes equations in unbounded 2D and 3D domains with the noise independent on ∇u were considered by Capiński and Peszat [15] and Brzeźniak and Peszat [12]. The solutions are constructed in weighted spaces. Invariant measures for stochastic Navier-Stokes equations with an additive noise in some unbounded 2D domains are investigated by Brzeźniak and Li [9]. Our results generalize the corresponding existence results from [15], [9] [21], and [31]. What concerns modelling of noise, we have tried to be as general as possible and include the roughest noise possible. One should bear in mind that the rougher the noise the closer the model is to reality. Moreover, Landau and Lifshitz their fundamental 1959 work [26, Chapter 17] proposed to study NSEs under additional stochastic small fluctuations. Consequently these authors considered the classical balance laws for mass, energy and momentum forced by a random noise, to describe the fluctuations, in particular local stresses and temperature, which are not

related to the gradient of the corresponding quantities. In [27, Chapter 12] the same authors found correlations for the random forcing by following the general theory of fluctuations. One of the requirements imposed on the noise is that it is either spatially uncorrelated or correlated as little as possible.

The present paper is organized as follows. In Section 2 we recall basic definitions and introduce some auxiliary operators. In Section 3 we are concerned with the compactness results. In Section 4, we formulate the Navier-Stokes problem as an abstract stochastic evolution equation. The main theorem about existence and construction of the martingale solution is in Section 5. Some auxiliary results connected with the proof are given in Appendices A and B. In Section 6, we consider some example of the noise. For the convenience of the reader, in Appendix C we recall Lemma 2.5 from [20] with the proof. Section 7 and Appendix D are devoted to 2D Navier-Stokes equations.

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2. FUNCTIONAL SETTING

2.1. Notations. Let $(X, |\cdot|_X), (Y, |\cdot|_Y)$ be two real normed spaces. The symbol $\mathcal{L}(X, Y)$ stands for the space of all bounded linear operators from X to Y . If $Y = \mathbb{R}$, then $X' := \mathcal{L}(X, \mathbb{R})$ is called the dual space of X . The symbol ${}_X\langle \cdot, \cdot \rangle_X$ denotes the standard duality pairing. If no confusion seems likely we omit the subscripts X', X and write $\langle \cdot, \cdot \rangle$. If both spaces X and Y are separable Hilbert, then by $\mathcal{T}_2(Y, X)$ we will denote the Hilbert space of all Hilbert-Schmidt operators from Y to X endowed with the standard norm.

Assume that X, Y are Banach spaces. Let A be a densely defined linear operator from X to Y and let $D(A)$ denote the domain of A . Let

$$D(A') := \{y' \in Y' : \text{the linear functional } y' \circ A : D(A) \rightarrow \mathbb{R} \text{ is bounded.}\}$$

Note that $D(A') = Y'$ if A is bounded. Let $y' \in D(A')$. Since A is densely defined, the functional $y' \circ A$ can be uniquely extended to the linear bounded functional $\overline{y' \circ A} \in X'$. The operator A' defined by

$$A'y' := \overline{y' \circ A}, \quad y' \in D(A'),$$

is called the dual operator of A .

Assume that X, Y are Hilbert spaces with scalar products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$, respectively. Let $A : X \supset D(A) \rightarrow Y$ be a densely defined linear operator.

By A^* we denote the adjoint operator of A . In particular, $D(A^*) \subset Y$, $A^* : D(A^*) \rightarrow X$ and

$$(Ax, y)_Y = (x, A^*y)_X, \quad x \in D(A), \quad y \in D(A^*).$$

Note that $D(A^*) = Y$ if A is bounded.

2.2. Basic definitions. Let $\mathcal{O} \subset \mathbb{R}^d$ be an open subset with smooth boundary $\partial\mathcal{O}$, $d = 2, 3$. Let $p \in (1, \infty)$ and let $L^p(\mathcal{O}, \mathbb{R}^d)$ denote the Banach space of Lebesgue measurable \mathbb{R}^d -valued p -th power integrable functions on the set \mathcal{O} . The norm in $L^p(\mathcal{O}, \mathbb{R}^d)$ is given by

$$\|u\|_{L^p} := \left(\int_{\mathcal{O}} |u(x)|^p dx \right)^{\frac{1}{p}}, \quad u \in L^p(\mathcal{O}, \mathbb{R}^d).$$

By $L^\infty(\mathcal{O}, \mathbb{R}^d)$ we denote the Banach space of Lebesgue measurable essentially bounded \mathbb{R}^d -valued functions defined on \mathcal{O} . The norm is given by

$$\|u\|_{L^\infty} := \text{esssup} \{|u(x)|, x \in \mathcal{O}\}, \quad u \in L^\infty(\mathcal{O}, \mathbb{R}^d).$$

If $p = 2$, then $L^2(\mathcal{O}, \mathbb{R}^d)$ is a Hilbert space with the scalar product given by

$$(u, v)_{L^2} := \int_{\mathcal{O}} u(x) \cdot v(x) dx, \quad u, v \in L^2(\mathcal{O}, \mathbb{R}^d).$$

Let $H^1(\mathcal{O}, \mathbb{R}^d)$ stand for the Sobolev space of all $u \in L^2(\mathcal{O}, \mathbb{R}^d)$ for which there exist weak derivatives $\frac{\partial u}{\partial x_i} \in L^2(\mathcal{O}, \mathbb{R}^d)$, $i = 1, 2, \dots, d$. It is a Hilbert space with the scalar product given by

$$(u, v)_{H^1} := (u, v)_{L^2} + ((u, v)), \quad u, v \in H^1(\mathcal{O}, \mathbb{R}^d),$$

where

$$((u, v)) := (\nabla u, \nabla v)_{L^2} = \sum_{i=1}^d \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} dx, \quad u, v \in H^1(\mathcal{O}, \mathbb{R}^d). \quad (2.1)$$

Let $\mathcal{C}_c^\infty(\mathcal{O}, \mathbb{R}^d)$ denote the space of all \mathbb{R}^d -valued functions of class \mathcal{C}^∞ with compact supports contained in \mathcal{O} and let

$$\begin{aligned} \mathcal{V} &:= \{u \in \mathcal{C}_c^\infty(\mathcal{O}, \mathbb{R}^d) : \text{div } u = 0\}, \\ H &:= \text{the closure of } \mathcal{V} \text{ in } L^2(\mathcal{O}, \mathbb{R}^d), \\ V &:= \text{the closure of } \mathcal{V} \text{ in } H^1(\mathcal{O}, \mathbb{R}^d). \end{aligned}$$

In the space H we consider the scalar product and the norm inherited from $L^2(\mathcal{O}, \mathbb{R}^d)$ and denote them by $(\cdot, \cdot)_H$ and $|\cdot|_H$, respectively, i.e.

$$(u, v)_H := (u, v)_{L^2}, \quad |u|_H := \|u\|_{L^2}, \quad u, v \in H.$$

In the space V we consider the scalar product inherited from $H^1(\mathcal{O}, \mathbb{R}^d)$, i.e.

$$(u, v)_V := (u, v)_H + ((u, v)), \quad (2.2)$$

where $((\cdot, \cdot))$ is defined in (2.1), and the norm the norm induced by $(\cdot, \cdot)_V$, i.e.

$$\|u\|_V^2 := |u|_H^2 + \|u\|^2, \quad (2.3)$$

where $\|u\|^2 := \|\nabla u\|_{L^2}^2$.

Let us consider the following tri-linear form

$$b(u, w, v) = \int_{\mathcal{O}} (u \cdot \nabla w) v \, dx. \quad (2.4)$$

We will recall the fundamental properties of the form b . Since usually one considers the bounded domain case we want to recall only those results that are valid in unbounded domains as well.

By the Sobolev embedding Theorem and the Hölder inequality, we obtain the following estimates

$$|b(u, w, v)| \leq \|u\|_{L^4} \|w\|_V \|v\|_{L^4} \quad (2.5)$$

$$\leq c \|u\|_V \|w\|_V \|v\|_V, \quad u, w, v \in V \quad (2.6)$$

for some positive constant c . Thus the form b is continuous on V , see also [35]. Moreover, if we define a bilinear map B by $B(u, w) := b(u, w, \cdot)$, then by inequality (2.6) we infer that $B(u, w) \in V'$ for all $u, w \in V$ and that the following inequality holds

$$|B(u, w)|_{V'} \leq c \|u\|_V \|w\|_V, \quad u, w \in V. \quad (2.7)$$

Moreover, the mapping $B : V \times V \rightarrow V'$ is bilinear and continuous.

Let us also recall the following properties of the form b , see Temam [35], Lemma II.1.3,

$$b(u, w, v) = -b(u, v, w), \quad u, w, v \in V. \quad (2.8)$$

In particular,

$$b(u, v, v) = 0 \quad u, v \in V. \quad (2.9)$$

Let us, for any $s > 0$ define the following standard scale of Hilbert spaces

$$V_s := \text{the closure of } \mathcal{V} \text{ in } H^s(\mathcal{O}, \mathbb{R}^d).$$

If $s > \frac{d}{2} + 1$ then by the Sobolev embedding Theorem,

$$H^{s-1}(\mathcal{O}, \mathbb{R}^d) \hookrightarrow \mathcal{C}_b(\mathcal{O}, \mathbb{R}^d) \hookrightarrow L^\infty(\mathcal{O}, \mathbb{R}^d).$$

Here $\mathcal{C}_b(\mathcal{O}, \mathbb{R}^d)$ denotes the space of continuous and bounded \mathbb{R}^d -valued functions defined on \mathcal{O} . If $u, w \in V$ and $v \in V_s$ with $s > \frac{d}{2} + 1$ then

$$|b(u, w, v)| = |b(u, v, w)| \leq \|u\|_{L^2} \|w\|_{L^2} \|\nabla v\|_{L^\infty} \leq c \|u\|_{L^2} \|w\|_{L^2} \|v\|_{V_s}$$

for some constant $c > 0$. Thus, b can be uniquely extended to the tri-linear form (denoted by the same letter)

$$b : H \times H \times V_s \rightarrow \mathbb{R}$$

and $|b(u, w, v)| \leq c\|u\|_{L^2}\|w\|_{L^2}\|v\|_{V_s}$ for $u, w \in H$ and $v \in V_s$. At the same time the operator B can be uniquely extended to a bounded bilinear operator

$$B : H \times H \rightarrow V'_s.$$

In particular, it satisfies the following estimate

$$|B(u, w)|_{V'_s} \leq c|u|_H|w|_H, \quad u, w \in H. \quad (2.10)$$

We will also use the following notation, $B(u) := B(u, u)$.

Lemma 2.1. *The map $B : V \rightarrow V'$ is locally Lipschitz continuous, i.e. for every $r > 0$ there exists a constant L_r such that*

$$|B(u) - B(\tilde{u})|_{V'} \leq L_r\|u - \tilde{u}\|_V, \quad u, \tilde{u} \in V, \quad \|u\|_V, \|\tilde{u}\|_V \leq r. \quad (2.11)$$

Proof. This is classical but for completeness we provide the proof. The assertion follows from the following estimates

$$\begin{aligned} |B(u, u) - B(\tilde{u}, \tilde{u})|_{V'} &\leq |B(u, u - \tilde{u})|_{V'} + |B(u - \tilde{u}, \tilde{u})|_{V'} \\ &\leq \|B\|(\|u\|_V + \|\tilde{u}\|_V)\|u - \tilde{u}\|_V \leq 2r\|B\| \cdot \|u - \tilde{u}\|_V. \end{aligned}$$

Thus the Lipschitz condition holds with $L_r = 2r\|B\|$, where $\|B\|$ stands for the norm of the bilinear map $B : V \times V \rightarrow V'$. The proof is thus complete. \square

2.3. Some operators. Consider the natural embedding $j : V \hookrightarrow H$ and its adjoint $j^* : H \rightarrow V$. Since the range of j is dense in H , the map j^* is one-to-one. Let us put

$$\begin{aligned} D(A) &:= j^*(H) \subset V, \\ Au &:= (j^*)^{-1}u, \quad u \in D(A). \end{aligned} \quad (2.12)$$

Notice that for all $u \in D(A)$ and $v \in V$

$$(Au, v)_H = (u, v)_V. \quad (2.13)$$

Indeed, this follows immediately from the following equalities

$$(Au|v)_H = ((j^*)^{-1}u|v)_H = ((j^*)^{-1}u|jv)_H = (j^*(j^*)^{-1}u, v)_V = (u, v)_V.$$

Let

$$\mathcal{A}u := ((u, \cdot)), \quad u \in V,$$

where $((\cdot, \cdot))$ is defined by (2.1). Let us notice that if $u \in V$, then $\mathcal{A}u \in V'$ and

$$|\mathcal{A}u|_{V'} \leq \|u\|. \quad (2.14)$$

Indeed, from (2.3) and the following inequalities

$$|((u, v))| \leq \|u\| \cdot \|v\| \leq \|u\|(\|v\|^2 + |v|_H^2)^{\frac{1}{2}} = \|u\| \cdot \|v\|_V, \quad v \in V,$$

it follows that $\mathcal{A}u \in V'$ and inequality (2.14) holds. Denoting by $\langle \cdot, \cdot \rangle$ the dual pairing between V and V' , i.e. $\langle \cdot, \cdot \rangle := {}_{V'}\langle \cdot, \cdot \rangle_V$, we have the following equality

$$\langle \mathcal{A}u, v \rangle = ((u, v)), \quad u, v \in V. \quad (2.15)$$

Lemma 2.2.

(a) For any $u \in D(A)$ and $v \in V$:

$$((A - I)u|v)_H = ((u, v)) = \langle \mathcal{A}u, v \rangle,$$

where I stands for the identity operator on H . In particular,

$$|\mathcal{A}u|_{V'} \leq |(A - I)u|_H. \quad (2.16)$$

(b) $D(A)$ is dense in H .

Proof. To prove assertion (a), let $u \in D(A)$ and $v \in V$. By (2.13), (2.2) and (2.15), we have

$$(Au|v)_H = (u, v)_V = (u, v)_H + ((u, v)) = (Iu, v)_H + \langle \mathcal{A}u, v \rangle.$$

Let us move to the proof of part (b). Since V is dense in H , it is sufficient to prove that $D(A)$ is dense in V . Let $w \in V$ be an arbitrary element orthogonal to $D(A)$ with respect to the scalar product in V . Then

$$(u, w)_V = 0 \quad \text{for } u \in D(A).$$

On the other hand, by (a) and (2.2), $(u, w)_V = (Au, w)_H$ for $u \in D(A)$. Hence $(Au, w)_H = 0$ for $u \in D(A)$. Since $A : D(A) \rightarrow H$ is onto, we infer that $w = 0$, which completes the proof. \square

Let us assume that $s > 1$. It is clear that V_s is dense in V and the embedding $j_s : V_s \hookrightarrow V$ is continuous. Then by Lemma C.1 in Appendix C, there exists a Hilbert space U such that $U \subset V_s$, U is dense in V_s and

$$\text{the natural embedding } \iota_s : U \hookrightarrow V_s \text{ is compact.} \quad (2.17)$$

Then we have

$$U \xhookrightarrow[\iota_s]{} V_s \xhookrightarrow[j_s]{} V \xhookrightarrow[j]{} H \cong H' \xhookrightarrow[j']{} V' \xhookrightarrow[j'_s]{} V'_s \xhookrightarrow[\iota'_s]{} U'. \quad (2.18)$$

Since the embedding ι_s is compact, ι'_s is compact as well. Consider the composition

$$\iota := j \circ j_s \circ \iota_s : U \hookrightarrow H$$

and its adjoint

$$\iota^* := (j \circ j_s \circ \iota_s)^* = \iota_s^* \circ j_s^* \circ j^* : H \rightarrow U.$$

Note that ι is compact and since the range of ι is dense in H , $\iota^* : H \rightarrow U$ is one-to-one. Let us put

$$\begin{aligned} D(L) &:= \iota^*(H) \subset U, \\ Lu &:= (\iota^*)^{-1}u, \quad u \in D(L). \end{aligned} \quad (2.19)$$

It is clear that $L : D(L) \rightarrow H$ is onto. Let us also notice that

$$(Lu, w)_H = (u, w)_U, \quad u \in D(L), \quad w \in U. \quad (2.20)$$

Indeed, by (2.19) we have for all $u \in D(L)$ and $w \in U$

$$(Lu, w)_H = ((\iota^*)^{-1}u, \iota w)_H = (\iota^*(\iota^*)^{-1}u, w)_U = (u, w)_U,$$

which proves (2.20). By equality (2.20) and the densiness of U in H , we infer similarly as in the proof of assertion (b) in Lemma 2.2 that $D(L)$ is dense in H .

Moreover, for $u \in D(L)$,

$$\begin{aligned} Lu &= (\iota^*)^{-1}u = (\iota_s^* \circ j_s^* \circ j^*)^{-1}u = (j^*)^{-1} \circ (j_s^*)^{-1} \circ (\iota_s^*)^{-1}u \\ &= A \circ (j_s^*)^{-1} \circ (\iota_s^*)^{-1}u, \end{aligned}$$

where A is defined by (2.12).

Let us also put

$$\begin{aligned} D(L_s) &:= \iota_s^*(V_s) \subset U, \\ L_s u &:= (\iota_s^*)^{-1}u, \quad u \in D(L_s) \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} D(A_s) &:= j_s^*(V) \subset V_s, \\ A_s u &:= (j_s^*)^{-1}u, \quad u \in D(A_s). \end{aligned} \quad (2.22)$$

The operators $L_s : D(L_s) \rightarrow V_s$ and $A_s : D(A_s) \rightarrow V$ are densely defined and onto. Let us also notice that

$$L = A \circ A_s \circ L_s. \quad (2.23)$$

Since L is self-adjoint and L^{-1} is compact, there exists an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of H composed of the eigenvectors of operator L . Let λ_i be the eigenvalue corresponding to e_i , i.e.

$$Le_i = \lambda_i e_i, \quad i \in \mathbb{N}. \quad (2.24)$$

Notice that $e_i \in U$, $i \in \mathbb{N}$, because $D(L) \subset U$. Let us fix $n \in \mathbb{N}$ and let $\text{span}\{e_1, \dots, e_n\}$ denote the linear space spanned by the vectors e_1, \dots, e_n . Let P_n be the operator from U' to $\text{span}\{e_1, \dots, e_n\}$ defined by

$$P_n u^* := \sum_{i=1}^n {}_{U'}\langle u^*, e_i \rangle_U e_i, \quad u^* \in U'. \quad (2.25)$$

We will consider the restriction of the operator P_n to the space H denoted still by P_n . More precisely, we have $H \hookrightarrow U'$, i.e. every element $u \in H$ induces a functional $u^* \in U'$ by the formula

$${}_{U'}\langle u^*, v \rangle_U := (u, v)_H, \quad v \in U.$$

Thus the restriction of P_n to H is given by

$$P_n u = \sum_{i=1}^n (u, e_i)_H e_i, \quad u \in H. \quad (2.26)$$

Hence in particular, P_n is the $(\cdot, \cdot)_H$ -orthogonal projection from H onto $\text{span}\{e_1, \dots, e_n\}$. Restrictions of P_n to other spaces considered in (2.18) will also be denoted by P_n .

Lemma 2.3. *The following equality holds*

$$(P_n u^*, v)_H = {}_{U'}\langle u^*, P_n v \rangle_U, \quad u^* \in U', \quad v \in U. \quad (2.27)$$

Proof. Let us fix $u^* \in U'$ and $v \in U$. By (2.25) and (2.26) the assertion follows from the following equalities

$$\begin{aligned} (P_n u^*, v)_H &= \left(\sum_{i=1}^n {}_{U'}\langle u^*, e_i \rangle_U e_i, v \right)_H = \sum_{i=1}^n {}_{U'}\langle u^*, e_i \rangle_U (v, e_i)_H \\ &= {}_{U'}\langle u^*, \sum_{i=1}^n (v, e_i)_H e_i \rangle_U = {}_{U'}\langle u^*, P_n v \rangle_U. \end{aligned}$$

The proof of (2.27) is thus complete. \square

Let us denote

$$\tilde{e}_i := \frac{e_i}{\|e_i\|_U}, \quad i \in \mathbb{N}.$$

Lemma 2.4. (a) *The system $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ is the orthonormal basis in the space $(U, (\cdot, \cdot)_U)$. Moreover,*

$$\lambda_i = \|e_i\|_U^2, \quad i \in \mathbb{N}. \quad (2.28)$$

(b) *For every $n \in \mathbb{N}$ and $u \in U$*

$$P_n u = \sum_{i=1}^n (u, \tilde{e}_i)_U \tilde{e}_i, \quad (2.29)$$

i.e. the restriction of P_n to U is the $(\cdot, \cdot)_U$ -orthogonal projection onto $\text{span}\{e_1, \dots, e_n\}$.

(c) *For every $u \in U$ and $s > 0$ we have*

- (i) $\lim_{n \rightarrow \infty} \|P_n u - u\|_U = 0,$
- (ii) $\lim_{n \rightarrow \infty} \|P_n u - u\|_{V_s} = 0,$
- (iii) $\lim_{n \rightarrow \infty} \|P_n u - u\|_V = 0.$

Proof. By (2.20) and (2.24)

$$(e_i, e_j)_U = (Le_i, e_j)_H = \lambda_i(e_i, e_j)_H = \lambda_i\delta_{ij}, \quad i, j \in \mathbb{N}, \quad (2.30)$$

where $\delta_{ij} := 1$ if $i = j$ and $\delta_{ij} := 0$ if $i \neq j$. In particular, equality (2.28) holds. By (2.30) and (2.28) the system $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ is orthonormal in U . To prove that it is a basis in U let $h \in U$ be an arbitrary vector $(\cdot, \cdot)_U$ -orthogonal to each e_i , $i \in \mathbb{N}$. By (2.20) and (2.24) we obtain the following equalities

$$0 = (h, e_i)_U = (h, Le_i)_H = \lambda_i(h, e_i)_H, \quad i \in \mathbb{N}.$$

Since $\lambda_i > 0$,

$$(h, e_i)_H = 0, \quad i \in \mathbb{N}. \quad (2.31)$$

Since $\{e_i\}_{i \in \mathbb{N}}$ is the $(\cdot, \cdot)_H$ -orthonormal basis in H , we infer that $h = 0$. This means that the system $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ is the $(\cdot, \cdot)_U$ -orthonormal basis in U .

To prove assertion (b) let us fix $u \in U$. By (2.28), (2.24) and (2.20), we have

$$(u, e_i)_H e_i = \left(u, \frac{\lambda_i e_i}{\|e_i\|_U}\right)_H \frac{e_i}{\|e_i\|_U} = \left(u, L \frac{e_i}{\|e_i\|_U}\right)_H \frac{e_i}{\|e_i\|_U} = (u, \tilde{e}_i)_U \tilde{e}_i, \quad i \in \mathbb{N}. \quad (2.32)$$

By (2.26) and (2.32)

$$P_n u = \sum_{i=1}^n (u, e_i)_H e_i = \sum_{i=1}^n (u, \tilde{e}_i)_U \tilde{e}_i, \quad n \in \mathbb{N}.$$

Since $\{\tilde{e}_i\}_{i \in \mathbb{N}}$ is the $(\cdot, \cdot)_U$ -orthonormal basis in U , the restriction of P_n to the space U is the $(\cdot, \cdot)_U$ -orthogonal projection onto $\text{span}\{e_1, \dots, e_n\}$.

Assertion (c) follows immediately from (b) and the continuity of the embeddings

$$U \hookrightarrow V_s \hookrightarrow V.$$

This completes the proof of the Lemma. \square

3. COMPACTNESS RESULTS.

3.1. Deterministic compactness criteria. Let us choose $s > \frac{d}{2} + 1$. We have the following sequence of Hilbert spaces

$$U \xhookrightarrow{\iota_s} V_s \hookrightarrow V \hookrightarrow H \cong H' \hookrightarrow V' \hookrightarrow V'_s \xhookrightarrow{\iota'_s} U', \quad (3.1)$$

with the embedding ι_s being compact. In particular,

$$H \cong H' \xhookrightarrow{\iota'} U' \quad (3.2)$$

and ι' is compact as well. Let $(\mathcal{O}_R)_{R \in \mathbb{N}}$ be a sequence of bounded open subsets of \mathcal{O} with regular boundaries $\partial\mathcal{O}_R$ such that $\mathcal{O}_R \subset \mathcal{O}_{R+1}$ and

$\bigcup_{R=1}^{\infty} \mathcal{O}_R = \mathcal{O}$. We will consider the following spaces of restrictions of functions defined on \mathcal{O} to subsets \mathcal{O}_R , i.e.

$$H_{\mathcal{O}_R} := \{u|_{\mathcal{O}_R}; u \in H\} \quad V_{\mathcal{O}_R} := \{v|_{\mathcal{O}_R}; v \in V\} \quad (3.3)$$

with appropriate scalar products and norms, i.e.

$$(u, v)_{H_{\mathcal{O}_R}} := \int_{\mathcal{O}_R} uv \, dx, \quad u, v \in H_{\mathcal{O}_R},$$

$$(u, v)_{V_{\mathcal{O}_R}} := \int_{\mathcal{O}_R} uv \, dx + \int_{\mathcal{O}_R} \nabla u \nabla v \, dx, \quad u, v \in V_{\mathcal{O}_R}$$

and $|u|_{H_{\mathcal{O}_R}}^2 := (u, u)_{H_{\mathcal{O}_R}}$ for $u \in H_{\mathcal{O}_R}$ and $\|u\|_{V_{\mathcal{O}_R}}^2 := (u, u)_{V_{\mathcal{O}_R}}$ for $u \in V_{\mathcal{O}_R}$. The symbols $H'_{\mathcal{O}_R}$ and $V'_{\mathcal{O}_R}$ will stand for the corresponding dual spaces.

Since the sets \mathcal{O}_R are bounded,

$$\text{the embeddings } V_{\mathcal{O}_R} \hookrightarrow H_{\mathcal{O}_R} \text{ are compact.} \quad (3.4)$$

Let us consider the following three functional spaces, analogous to those considered in [31]:

$$\begin{aligned} \mathcal{C}([0, T], U') &:= \text{the space of continuous functions } u : [0, T] \rightarrow U' \\ &\text{with the topology } \mathcal{T}_1 \text{ induced by the norm} \\ \|u\|_{\mathcal{C}([0, T], U')} &:= \sup_{t \in [0, T]} |u(t)|_{U'}, \end{aligned}$$

$$\begin{aligned} L_w^2(0, T; V) &:= \text{the space } L^2(0, T; V) \text{ with the weak topology } \mathcal{T}_2, \\ L^2(0, T; H_{loc}) &:= \text{the space of measurable functions } u : [0, T] \rightarrow H \\ &\text{such that for all } R \in \mathbb{N} \end{aligned} \quad (3.5)$$

$$q_{T,R}(u) := \|u\|_{L^2(0, T; H_{\mathcal{O}_R})} := \left(\int_0^T \int_{\mathcal{O}_R} |u(t, x)|^2 \, dx dt \right)^{\frac{1}{2}} < \infty, \quad (3.6)$$

with the topology \mathcal{T}_3 generated by the seminorms $(q_{T,R})_{R \in \mathbb{N}}$.

The following lemma was inspired by the classical Dubinsky Theorem, see for instance the monograph [38], and the compactness result due to Mikulevicius and Rozovskii contained in [31, Lemma 2.7]. The proof will be a certain modification of the proof of [38, Theorem IV.4.1], see also [28].

Lemma 3.1. *Let*

$$\tilde{\mathcal{Z}} := \mathcal{C}([0, T]; U') \cap L_w^2(0, T; V) \cap L^2(0, T; H_{loc}) \quad (3.7)$$

and let $\tilde{\mathcal{T}}$ be the supremum of the corresponding topologies. Then a set $\mathcal{K} \subset \tilde{\mathcal{Z}}$ is $\tilde{\mathcal{T}}$ -relatively compact if the following two conditions hold

- (i) $\sup_{u \in \mathcal{K}} \int_0^T \|u(s)\|_V^2 \, ds < \infty$, i.e. \mathcal{K} is bounded in $L^2(0, T; V)$,
- (ii) $\lim_{\delta \rightarrow 0} \sup_{u \in \mathcal{K}} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \delta}} |u(t) - u(s)|_{U'} = 0$.

Proof. We can assume that \mathcal{K} is closed in $\tilde{\mathcal{T}}$. Because of the assumption (i), the restriction to \mathcal{K} of the weak topology in $L_w^2(0, T; V)$ is metrizable. Since the topology in $L^2(0, T; H_{loc})$ is defined by the countable family of seminorms (3.6), this space is also metrizable. Thus the compactness of a subset of $\tilde{\mathcal{Z}}$ is equivalent to its sequential compactness. Let (u_n) be a sequence in \mathcal{K} . By the Banach-Alaoglu Theorem condition (i) yields that \mathcal{K} is compact in $L_w^2(0, T; V)$.

Arguing analogously to the proof of the classical Arzelà-Ascoli Theorem, we will prove that (u_n) is compact in $\mathcal{C}([0, T]; U')$.

Let us consider the following set

$$I_\infty := \{t \in [0, T] : \lim_{n \rightarrow \infty} \|u_n(t)\|_V = \infty\}. \quad (3.8)$$

The set I_∞ is Lebesgue measurable because

$$I_\infty = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcap_{l=k}^{\infty} \{\|u_l(t)\|_V^2 \geq n\}.$$

Moreover, its measure is equal to zero. Indeed, let us notice that otherwise

$$\int_0^T \|u_n(t)\|_V^2 dt \geq \int_{I_\infty} \|u_n(t)\|_V^2 dt \geq n \text{Leb}(I_\infty) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which contradicts assumption (i).

By (3.8), for every $t \in [0, T] \setminus I_\infty$, the sequence $(u_n(t))_{n \in \mathbb{N}}$ contains a subsequence bounded in V . Furthermore, since the embedding $V \hookrightarrow U'$ is compact, this subsequence contains a subsequence convergent in U' .

Let $\{t_i\}_{i \in \mathbb{N}} \subset [0, T] \setminus I_\infty$ be a dense subset of $[0, T]$. Using the diagonal method we can choose a subsequence, still denoted by $(u_n)_{n \in \mathbb{N}}$ such that

$$(u_n(t_i))_{n \in \mathbb{N}} \text{ is convergent in } U' \text{ for each } i \in \mathbb{N}. \quad (3.9)$$

We will prove that the sequence (u_n) is Cauchy in $\mathcal{C}([0, T]; U')$. To this end let us fix $\varepsilon > 0$. By (ii) there exists $\delta > 0$ such that

$$\sup_{v \in \mathcal{K}} \sup_{\substack{s, t \in [0, T], \\ |s-t| \leq \delta}} |v(t) - v(s)|_{U'} < \frac{\varepsilon}{3}.$$

Let us fix $t \in [0, T]$. There exists $i \in \mathbb{N}$ such that $|t - t_i| \leq \delta$. Then for sufficiently large $m, n \in \mathbb{N}$ we have the following estimates

$$|u_n(t) - u_m(t)|_{U'} \leq |u_n(t) - u_n(t_i)|_{U'} + |u_n(t_i) - u_m(t_i)|_{U'} + |u_m(t_i) - u_m(t)|_{U'} \leq \varepsilon.$$

Since $t \in [0, T]$ was chosen in an arbitrary way we infer that

$$\sup_{t \in [0, T]} |u_n(t) - u_m(t)|_{U'} \leq \varepsilon$$

which means that the sequence (u_n) is Cauchy in $\mathcal{C}([0, T]; U')$.

Therefore there exists a subsequence $(u_{n_k}) \subset (u_n)$ and $u \in L^2(0, T; V) \cap \mathcal{C}([0, T]; U')$ such that

$$u_{n_k} \rightarrow u \quad \text{in } L^2_w(0, T; V) \cap \mathcal{C}([0, T]; U') \quad \text{as } k \rightarrow \infty.$$

In particular, for all $p \in [1, \infty)$

$$u_{n_k} \rightarrow u \quad \text{in } L^p(0, T; U') \quad \text{as } k \rightarrow \infty.$$

We claim that

$$u_{n_k} \rightarrow u \quad \text{in } L^2(0, T; H_{loc}) \quad \text{as } k \rightarrow \infty. \quad (3.10)$$

In order to prove (3.10) let us fix $R > 0$. Since, by (3.4) the embedding $V_{\mathcal{O}_R} \hookrightarrow H_{\mathcal{O}_R}$ is compact and the embeddings $H_{\mathcal{O}_R} \hookrightarrow H' \hookrightarrow U'$ are continuous, by the Lions Lemma, [28], for every $\varepsilon > 0$ there exists a constant $C = C_{\varepsilon, R} > 0$ such that

$$|u|_{H_{\mathcal{O}_R}}^2 \leq \varepsilon \|u\|_{V_{\mathcal{O}_R}}^2 + C|u|_{U'}^2, \quad u \in V.$$

Thus for almost all $s \in [0, T]$

$$|u_{n_k}(s) - u(s)|_{H_{\mathcal{O}_R}}^2 \leq \varepsilon \|u_{n_k}(s) - u(s)\|_{V_{\mathcal{O}_R}}^2 + C|u_{n_k}(s) - u(s)|_{U'}^2, \quad k \in \mathbb{N},$$

and so

$$\|u_{n_k} - u\|_{L^2(0, T; H_{\mathcal{O}_R})}^2 \leq \varepsilon \|u_{n_k} - u\|_{L^2(0, T; V_{\mathcal{O}_R})}^2 + C\|u_{n_k} - u\|_{L^2(0, T; U')}^2, \quad k \in \mathbb{N}.$$

Passing to the upper limit as $k \rightarrow \infty$ in the above inequality and using the estimate

$$\|u_{n_k} - u\|_{L^2(0, T; V_{\mathcal{O}_R})}^2 \leq 2(\|u_{n_k}\|_{L^2(0, T; V_{\mathcal{O}_R})}^2 + \|u\|_{L^2(0, T; V_{\mathcal{O}_R})}^2) \leq 4c_2,$$

where $c_2 = \sup_{u \in \mathcal{K}} \|u\|_{L^2(0, T; V)}^2$, we infer that

$$\limsup_{k \rightarrow \infty} \|u_{n_k} - u\|_{L^2(0, T; H_{\mathcal{O}_R})}^2 \leq 4c_2\varepsilon,$$

By the arbitrariness of ε ,

$$\lim_{k \rightarrow \infty} \|u_{n_k} - u\|_{L^2(0, T; H_{\mathcal{O}_R})}^2 = 0.$$

The proof of Lemma is thus complete. \square

Let H_w denote the Hilbert space H endowed with the weak topology. Let

$$\begin{aligned} \mathcal{C}([0, T]; H_w) := & \text{ the space of weakly continuous functions } u : [0, T] \rightarrow H \\ & \text{ endowed with the weakest topology } \mathcal{T}_4 \text{ such that} \\ & \text{ for all } h \in H \text{ the mappings} \\ \mathcal{C}([0, T]; H_w) \ni u \mapsto & (u(\cdot), h)_H \in \mathcal{C}([0, T]; \mathbb{R}) \\ & \text{ are continuous.} \end{aligned} \quad (3.11)$$

In particular, $u_n \rightarrow u$ in $\mathcal{C}([0, T]; H_w)$ iff for all $h \in H$:

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |(u_n(t) - u(t)|h)_H| = 0.$$

Let us consider the ball

$$\mathbb{B} := \{x \in H : |x|_H \leq r\}.$$

Let \mathbb{B}_w denote the ball \mathbb{B} endowed with the weak topology. It is well-known that \mathbb{B}_w is metrizable, see [6]. Let q denote the metric compatible with the weak topology on \mathbb{B} . Let us consider the following subspace of the space $\mathcal{C}([0, T]; H_w)$

$$\mathcal{C}([0, T]; \mathbb{B}_w) = \{u \in \mathcal{C}([0, T]; H_w) : \sup_{t \in [0, T]} |u(t)|_H \leq r\}. \quad (3.12)$$

The space $\mathcal{C}([0, T]; \mathbb{B}_w)$ is metrizable with

$$\varrho(u, v) = \sup_{t \in [0, T]} q(u(t), v(t)). \quad (3.13)$$

Since by the Banach-Alaoglu Theorem \mathbb{B}_w is compact, $(\mathcal{C}([0, T]; \mathbb{B}_w), \varrho)$ is a complete metric space. The following lemma says that any sequence $(u_n) \subset \mathcal{C}([0, T]; \mathbb{B})$ convergent in $\mathcal{C}([0, T]; U')$ is also convergent in the space $\mathcal{C}([0, T]; \mathbb{B}_w)$.

Lemma 3.2. (see Lemma 2.1 in [10]) *Let $u_n : [0, T] \rightarrow H$, $n \in \mathbb{N}$ be functions such that*

- (i) $\sup_{n \in \mathbb{N}} \sup_{s \in [0, T]} |u_n(s)|_H \leq r$,
- (ii) $u_n \rightarrow u$ in $\mathcal{C}([0, T]; U')$.

Then $u, u_n \in \mathcal{C}([0, T]; \mathbb{B}_w)$ and $u_n \rightarrow u$ in $\mathcal{C}([0, T]; \mathbb{B}_w)$ as $n \rightarrow \infty$.

The proof of Lemma 3.2 is the same as the proof of Lemma 2.1 in [10].

The following lemma is essentially due to Mikulevicius and Rozovskii, see [31]. We prove a version of it which we will use in the forthcoming tightness criterion for the set of measures induced on the space \mathcal{Z} , see Corollary 3.9. The main difference in comparison to [31] is that instead of the whole space \mathbb{R}^d we consider any open subset \mathcal{O} and instead of the Fréchet space $H_{loc}^{-k_0}(\mathbb{R}^d)$ we take the Hilbert space U' .

Lemma 3.3. (see Lemma 2.7 in [31]) *Let*

$$\mathcal{Z} := \mathcal{C}([0, T]; U') \cap L_w^2(0, T; V) \cap L^2(0, T; H_{loc}) \cap \mathcal{C}([0, T]; H_w) \quad (3.14)$$

and let \mathcal{T} be the supremum of the corresponding topologies. Then a set $\mathcal{K} \subset \mathcal{Z}$ is \mathcal{T} -relatively compact if the following three conditions hold

- (a) $\sup_{u \in \mathcal{K}} \sup_{s \in [0, T]} |u(s)|_H < \infty$,
- (b) $\sup_{u \in \mathcal{K}} \int_0^T \|u(s)\|^2 ds < \infty$,
- (c) $\lim_{\delta \rightarrow 0} \sup_{u \in \mathcal{K}} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \delta}} |u(t) - u(s)|_{U'} = 0$.

Proof. Let us notice that $\mathcal{Z} = \tilde{\mathcal{Z}} \cap \mathcal{C}([0, T], H_w)$, where $\tilde{\mathcal{Z}}$ is defined by (3.7). Without loss of generality we can assume that \mathcal{K} is a closed subset of \mathcal{Z} . Because of the assumption (a) we may consider the metric subspace $\mathcal{C}([0, T]; \mathbb{B}_w) \subset \mathcal{C}([0, T]; H_w)$ defined by (3.12) and (3.13). From assumptions (a), (b) and definition (2.3) of the norm in V , it follows that the set \mathcal{K} is bounded in $L^2(0, T; V)$. Therefore the restrictions to \mathcal{K} of the four topologies considered in \mathcal{Z} are metrizable. Let (u_n) be a sequence in \mathcal{Z} . By Lemma 3.1 the boundedness of the set \mathcal{K} in $L^2(0, T; V)$ and assumption (c) imply that \mathcal{K} is compact in $\tilde{\mathcal{Z}}$. Hence in particular, there exists a subsequence, still denoted by (u_n) , convergent in $\mathcal{C}([0, T]; U')$. Thus by Lemma 3.2 the sequence (u_n) is convergent in $\mathcal{C}([0, T]; \mathbb{B}_w)$ as well. The proof of Lemma is thus complete. \square

3.2. Tightness criterion. Let (\mathbb{S}, ϱ) be a separable and complete metric space.

Definition 3.4. Let $u \in \mathcal{C}([0, T], \mathbb{S})$. The modulus of continuity of u on $[0, T]$ is defined by

$$m(u, \delta) := \sup_{s, t \in [0, T], |t-s| \leq \delta} \varrho(u(t), u(s)), \quad \delta > 0.$$

Let $(\Omega, \mathcal{F}, \mathbb{P},)$ be a probability space with filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions, see [29], and let $(X_n)_{n \in \mathbb{N}}$ be a sequence of continuous \mathbb{F} -adapted \mathbb{S} -valued processes.

Definition 3.5. We say that the sequence (X_n) of \mathbb{S} -valued random variables satisfies condition $[\tilde{\mathbf{T}}]$ iff $\forall \varepsilon > 0 \quad \forall \eta > 0 \quad \exists \delta > 0$:

$$\sup_{n \in \mathbb{N}} \mathbb{P}\{m(X_n, \delta) > \eta\} \leq \varepsilon. \quad (3.15)$$

Lemma 3.6. Assume that (X_n) satisfies condition $[\tilde{\mathbf{T}}]$. Let \mathbb{P}_n be the law of X_n on $\mathcal{C}([0, T], \mathbb{S})$, $n \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists a subset $A_\varepsilon \subset \mathcal{C}([0, T], \mathbb{S})$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n(A_\varepsilon) \geq 1 - \varepsilon$$

and

$$\lim_{\delta \rightarrow 0} \sup_{u \in A_\varepsilon} m(u, \delta) = 0. \quad (3.16)$$

Now, we recall the Aldous condition which is connected with condition $[\tilde{\mathbf{T}}]$, see [30] and [2]. This condition allows to investigate the modulus of continuity for the sequence of stochastic processes by means of stopped processes.

Definition 3.7. A sequence $(X_n)_{n \in \mathbb{N}}$ satisfies condition $[\mathbf{A}]$ iff

$\forall \varepsilon > 0 \quad \forall \eta > 0 \quad \exists \delta > 0$ such that for every sequence $(\tau_n)_{n \in \mathbb{N}}$ of \mathbb{F} -stopping times with $\tau_n \leq T$ one has

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} \mathbb{P}\{\varrho(X_n(\tau_n + \theta), X_n(\tau_n)) \geq \eta\} \leq \varepsilon.$$

Lemma 3.8. (See [30], Th. 3.2 p. 29) *Conditions [A] and $[\tilde{\mathbf{T}}]$ are equivalent.*

Using the deterministic compactness criterion formulated in Lemma 3.3 we obtain the following corollary which we will use to prove tightness of the laws defined by the Galerkin approximations. Let us first recall that U is a Hilbert space such that

$$U \hookrightarrow V \hookrightarrow H$$

and the embedding $U \hookrightarrow V$ is dense and compact. Moreover, we consider the following space

$$\mathcal{Z} := \mathcal{C}([0, T]; U') \cap L_w^2(0, T; V) \cap L^2(0, T; H_{loc}) \cap \mathcal{C}([0, T]; H_w),$$

equipped with the topology \mathcal{T} , see (3.14).

Corollary 3.9. (tightness criterion) *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of continuous \mathbb{F} -adapted U' -valued processes such that*

(a) *there exists a positive constant C_1 such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\sup_{s \in [0, T]} |X_n(s)|_H^2 \right] \leq C_1.$$

(b) *there exists a positive constant C_2 such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\int_0^T \|X_n(s)\|^2 ds \right] \leq C_2.$$

(c) *$(X_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition [A] in U' .*

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on \mathcal{Z} . Then for every $\varepsilon > 0$ there exists a compact subset K_ε of \mathcal{Z} such that

$$\sup_{n \in \mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

Proof. Let $\varepsilon > 0$. By the Chebyshev inequality and (a), we infer that for any $n \in \mathbb{N}$ and any $r > 0$

$$\mathbb{P} \left(\sup_{s \in [0, T]} |X_n(s)|_H^2 > r \right) \leq \frac{\mathbb{E} [\sup_{s \in [0, T]} |X_n(s)|_H^2]}{r} \leq \frac{C_1}{r}.$$

Let R_1 be such that $\frac{C_1}{R_1} \leq \frac{\varepsilon}{3}$. Then

$$\sup_{n \in \mathbb{N}} \mathbb{P} \left(\sup_{s \in [0, T]} |X_n(s)|_H^2 > R_1 \right) \leq \frac{\varepsilon}{3}.$$

Let $B_1 := \{u \in \mathcal{Z} : \sup_{s \in [0, T]} |u(s)|_H^2 \leq R_1\}$.

By the Chebyshev inequality and (b), we infer that for any $n \in \mathbb{N}$ and any $r > 0$

$$\mathbb{P}(\|X_n\|_{L^2(0,T;V)} > r) \leq \frac{\mathbb{E}[\|X_n\|_{L^2(0,T;V)}^2]}{r^2} \leq \frac{C_2}{r^2}.$$

Let R_2 be such that $\frac{C_2}{R_2^2} \leq \frac{\varepsilon}{3}$. Then

$$\sup_{n \in \mathbb{N}} \mathbb{P}(\|X_n\|_{L^2(0,T;V)} > R_2) \leq \frac{\varepsilon}{3}.$$

Let $B_2 := \{u \in \mathcal{Z} : \|u\|_{L^2(0,T;V)} \leq R_2\}$.

By Lemmas 3.8 and 3.6 there exists a subset $A_{\frac{\varepsilon}{3}} \subset \mathcal{C}([0,T], U')$ such that $\tilde{\mathbb{P}}_n(A_{\frac{\varepsilon}{3}}) \geq 1 - \frac{\varepsilon}{3}$ and

$$\lim_{\delta \rightarrow 0} \sup_{u \in A_{\frac{\varepsilon}{3}}} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \delta}} |u(t) - u(s)|_{U'} = 0.$$

It is sufficient to define K_ε as the closure of the set $B_1 \cap B_2 \cap A_{\frac{\varepsilon}{3}}$ in \mathcal{Z} . By Lemma 3.3, K_ε is compact in \mathcal{Z} . The proof is thus complete. \square

Digression. If we omit assumption (a) in Corollary 3.9, then using Lemma 3.1, we obtain the following tightness criterion in the space $(\tilde{\mathcal{Z}}, \tilde{\mathcal{T}})$ defined by (3.7).

Corollary 3.10. (tightness criterion in $\tilde{\mathcal{Z}}$) *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of continuous \mathbb{F} -adapted U' -valued processes such that*

(i) *there exists a positive constant C such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\int_0^T \|X_n(s)\|_V^2 ds \right] \leq C.$$

(ii) *$(X_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition $[\mathbf{A}]$ in U' .*

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on $\tilde{\mathcal{Z}}$. Then for every $\varepsilon > 0$ there exists a compact subset K_ε of $\tilde{\mathcal{Z}}$ such that

$$\sup_{n \in \mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

3.3. The Skorokhod Theorem. We will use the following Jakubowski's version of the Skorokhod Theorem in the form given by Brzeźniak and Ondreját [11], see also [23].

Theorem 3.11. (Theorem A.1 in [11]) *Let \mathcal{X} be a topological space such that there exists a sequence $\{f_m\}$ of continuous functions $f_m : \mathcal{X} \rightarrow \mathbb{R}$ that separates points of \mathcal{X} . Let us denote by \mathcal{S} the σ -algebra generated by the maps $\{f_m\}$. Then*

- (j1) *every compact subset of \mathcal{X} is metrizable,*
- (j2) *if (μ_m) is a tight sequence of probability measures on $(\mathcal{X}, \mathcal{S})$, then there exists a subsequence (m_k) , a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with \mathcal{X} -valued Borel measurable variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω . Moreover, the law of ξ is a Radon measure.*

Using Theorem 3.11, we obtain the following corollary which we will apply to construct a martingale solution to the Navier-Stokes equations.

Corollary 3.12. *Let $(\eta_n)_{n \in \mathbb{N}}$ be a sequence of \mathcal{Z} -valued random variables such that their laws $\mathcal{L}(\eta_n)$ on $(\mathcal{Z}, \mathcal{T})$ form a tight sequence of probability measures. Then there exists a subsequence (n_k) , a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and \mathcal{Z} -valued random variables $\tilde{\eta}, \tilde{\eta}_k$, $k \in \mathbb{N}$ such that the variables η_k and $\tilde{\eta}_k$ have the same laws on \mathcal{Z} and $\tilde{\eta}_k$ converges to $\tilde{\eta}$ almost surely on $\tilde{\Omega}$.*

Proof. It is sufficient to prove that on each space appearing in the definition (3.14) of the space \mathcal{Z} there exists a countable set of continuous real-valued functions separating points.

Since the spaces $\mathcal{C}([0, T]; U')$ and $L^2(0, T; H_{loc})$ are separable metrizable and complete, this condition is satisfied, see [3], exposé 8.

For the space $L_w^2(0, T; V)$ it is sufficient to put

$$f_m(u) := \int_0^T (u(t), v_m(t))_V dt \in \mathbb{R}, \quad u \in L_w^2(0, T; V), \quad m \in \mathbb{N},$$

where $\{v_m, m \in \mathbb{N}\}$ is a dense subset of $L^2(0, T; V)$.

Let us consider the space $\mathcal{C}([0, T]; H_w)$ defined by (3.11). Let $\{h_m, m \in \mathbb{N}\}$ be any dense subset of H and let Q_T be the set of rational numbers belonging to the interval $[0, T]$. Then the family $\{f_{m,t}, m \in \mathbb{N}, t \in Q_T\}$ defined by

$$f_{m,t}(u) := (u(t), h_m)_H \in \mathbb{R}, \quad u \in \mathcal{C}([0, T]; H_w), \quad m \in \mathbb{N}, \quad t \in Q_T$$

consists of continuous functions separating points in $\mathcal{C}([0, T]; H_w)$. Now, the statement follows from Theorem 3.11, which completes the proof. \square

4. STOCHASTIC NAVIER-STOKES EQUATIONS

We consider the following stochastic evolution equation

$$\begin{cases} du(t) + \mathcal{A}u(t) dt + B(u(t), u(t)) dt = f(t) dt + G(u(t)) dW(t), & t \in [0, T], \\ u(0) = u_0. \end{cases} \quad (4.1)$$

Assumptions. We assume that

(A.1) $W(t)$ is a cylindrical Wiener process in a separable Hilbert space Y defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$;

(A.2) $u_0 \in H$, $f \in L^2(0, T; V')$;

(A.3) The mapping $G : V \rightarrow \mathcal{T}_2(Y, H)$ is Lipschitz continuous and

$$2\langle \mathcal{A}u, u \rangle - \|G(u)\|_{\mathcal{T}_2(Y, H)}^2 \geq \eta \|u\|^2 - \lambda_0 |u|_H^2 - \rho, \quad u \in V, \quad (\text{G})$$

for some constants λ_0, ρ and $\eta \in (0, 2]$.

Moreover, G extends to a mapping $G : H \rightarrow \mathcal{T}_2(Y, V')$ such that

$$\|G(u)\|_{\mathcal{T}_2(Y, V')}^2 \leq C(1 + |u|_H^2), \quad u \in H. \quad (\text{G}^*)$$

for some $C > 0$. Moreover, for every $\psi \in \mathcal{V}$

$$\text{the mapping } H \ni u \mapsto \langle G(u), \psi \rangle \in Y \text{ is continuous,} \quad (\text{G}^{**})$$

if in the space H we consider the Fréchet topology inherited from the space $L_{loc}^2(\mathcal{O}, \mathbb{R}^d)$.

By $L_{loc}^2(\mathcal{O}, \mathbb{R}^d)$ we denote the space of all Lebesgue measurable \mathbb{R}^d -valued functions v such that $\int_K |v(x)|^2 dx < \infty$ for every compact subset $K \subset \mathcal{O}$. In this space we consider the Fréchet topology generated by the family of seminorms

$$\left(\int_{\mathcal{O}_R} |v(x)|^2 dx \right)^{\frac{1}{2}}, \quad R \in \mathbb{N},$$

where $(\mathcal{O}_R)_{R \in \mathbb{N}}$ is any increasing sequence of open bounded subsets of \mathcal{O} .

More precisely, in condition (G^{**}) we identify $\langle G(\cdot), \psi \rangle$ with the mapping $\psi^{**}G : H \rightarrow Y'$ defined by

$$(\psi^{**}G(u))y := (G(u)y)\psi \in \mathbb{R}, \quad u \in H, \quad y \in Y. \quad (\text{G}'^{**})$$

Inequality (G) in assumption (A.3) is the same as considered by Flandoli and Gatarek in [21] for bounded domains. The assumption $\eta = 2$ corresponds to the case when the noise term does not depend on ∇u . We will prove that the set of measures induced on appropriate space by the solutions of the Galerkin equations is tight provided that assumptions (G) and (G^*) are satisfied. Assumptions (G^*) , (G^{**}) will be important in passing to the limit

as $n \rightarrow \infty$ in the Galerkin approximation. Assumption (G**) is essential if the domain is unbounded.

Definition 4.1. We say that there exists a **martingale solution** of the equation (4.1) iff there exist

- a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ with filtration $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_t\}_{t \geq 0}$,
- a cylindrical Wiener process \hat{W} on the space Y ,
- and a progressively measurable process $u : [0, T] \times \Omega \rightarrow H$ with $\hat{\mathbb{P}}$ -a.e. paths

$$u(\cdot, \omega) \in \mathcal{C}([0, T], H_w) \cap L^2(0, T; V)$$

such that for all $t \in [0, T]$ and all $v \in \mathcal{V}$:

$$\begin{aligned} (u(t), v)_H &+ \int_0^t \langle \mathcal{A}u(s), v \rangle ds + \int_0^t \langle B(u(s), u(s)), v \rangle ds \\ &= (u_0, v)_H + \int_0^t \langle f(s), v \rangle ds + \left\langle \int_0^t G(u(s)) d\hat{W}(s), v \right\rangle \end{aligned} \quad (4.2)$$

the identity holds $\hat{\mathbb{P}}$ -a.s.

5. EXISTENCE OF SOLUTIONS

Theorem 5.1. *Let assumptions (A.1)-(A.3) be satisfied. Then there exists a martingale solution $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, u)$ of problem (4.1) such that*

$$\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |u(t)|_H^2 + \int_0^T \|u(t)\|^2 dt \right] < \infty. \quad (5.1)$$

5.1. Faedo-Galerkin approximation. Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis in H composed of eigenvectors of L . Let $H_n := \text{span}\{e_1, \dots, e_n\}$ be the subspace with the norm inherited from H and let $P_n : U' \rightarrow H_n$ be defined by (2.25). Consider the following mapping

$$B_n(u) := P_n B(\chi_n(u), u), \quad u \in H_n,$$

where $\chi_n : H \rightarrow H$ is defined by $\chi_n(u) = \theta_n(|u|_{U'})u$ with $\theta_n : \mathbb{R} \rightarrow [0, 1]$ of class \mathcal{C}^∞ such that

$$\theta_n(r) = \begin{cases} 1 & \text{if } r \leq n \\ 0 & \text{if } r \geq n+1. \end{cases}$$

Since $H_n \subset H$, B_n is well defined. Moreover, $B_n : H_n \rightarrow H_n$ is globally Lipschitz continuous.

Let us consider the classical Faedo-Galerkin approximation in the space H_n

$$\begin{cases} du_n(t) = -[P_n \mathcal{A}u_n(t) + B_n(u_n(t)) - P_n f(t)] dt + P_n G(u_n(t)) dW(t), & t \in [0, T], \\ u_n(0) = P_n u_0. \end{cases} \quad (5.2)$$

The proof of the next result is standard and thus omitted.

Lemma 5.2. *For each $n \in \mathbb{N}$, there exists a solution of the Galerkin equation (5.2). Moreover, $u_n \in \mathcal{C}([0, T]; H_n)$, \mathbb{P} -a.s. and $\mathbb{E}[\int_0^T |u_n(s)|_H^q ds] < \infty$ for any $q \in [2, \infty)$.*

Using the Itô formula and the Burkholder-Davis-Gundy inequality, see [17], we will prove the following lemma about *a priori* estimates of the solutions u_n of (5.2). Let us put the following condition on p

$$\begin{cases} p \in [2, 2 + \frac{\eta}{2-\eta}) & \text{if } \eta \in (0, 2), \\ p \in [2, \infty) & \text{if } \eta = 2. \end{cases} \quad (5.3)$$

Lemma 5.3. *The processes $(u_n)_{n \in \mathbb{N}}$ satisfy the following estimates.*

- (i) *For every p satisfying (5.3) there exist positive constants $C_1(p)$ and $C_2(p)$ such that*

$$\sup_{n \geq 1} \mathbb{E} \left(\sup_{0 \leq s \leq T} |u_n(s)|_H^p \right) \leq C_1(p). \quad (5.4)$$

and

$$\sup_{n \geq 1} \mathbb{E} \left[\int_0^T |u_n(s)|_H^{p-2} \|u_n(s)\|^2 ds \right] \leq C_2(p). \quad (5.5)$$

- (ii) *In particular, with $C_2 := C_2(2)$*

$$\sup_{n \geq 1} \mathbb{E} \left[\int_0^T \|u_n(s)\|^2 ds \right] \leq C_2. \quad (5.6)$$

Proof. See Appendix A. □

5.2. Tightness. For each $n \in \mathbb{N}$, the solution u_n of the Galerkin equation defines a measure $\mathcal{L}(u_n)$ on $(\mathcal{Z}, \mathcal{T})$. Using Corollary 3.9, inequality (5.6) and inequality (5.4) with $p = 2$ we will prove the tightness of this set of measures.

Lemma 5.4. *The set of measures $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$ is tight on $(\mathcal{Z}, \mathcal{T})$.*

Proof. We apply Corollary 3.9. According to estimates (5.4) and (5.6), conditions (a), (b) are satisfied. Thus, it is sufficient to prove that the sequence $(u_n)_{n \in \mathbb{N}}$ satisfies the Aldous condition [A]. Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of stopping times such that $0 \leq \tau_n \leq T$. By (5.2), we have

$$\begin{aligned} u_n(t) &= P_n u_0 - \int_0^t P_n \mathcal{A} u_n(s) ds - \int_0^t B_n(u_n(s)) ds + \int_0^t P_n f(s) ds \\ &\quad + \int_0^t P_n G(u_n(s)) dW(s) \\ &=: J_1^n(t) + J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t), \quad t \in [0, T]. \end{aligned}$$

Let $\theta > 0$. First, we make some estimates for each term of the above equality.

Ad. J_2^n . Since $\mathcal{A} : V \rightarrow V'$ and $|\mathcal{A}(u)|_{V'} \leq \|u\|$ and the embedding $V' \hookrightarrow U'$ is continuous, then by the Hölder inequality and (5.6), we have the following estimates

$$\begin{aligned} \mathbb{E}[|J_2^n(\tau_n + \theta) - J_2^n(\tau_n)|_{U'}] &= \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n + \theta} P_n \mathcal{A} u_n(s) ds\right|_{U'}\right] \\ &\leq c \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} |\mathcal{A} u_n(s)|_{V'} ds\right] \leq c \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\| ds\right] \\ &\leq c \theta^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T \|u_n(s)\|^2 ds\right]\right)^{\frac{1}{2}} \leq c C_2 \cdot \theta^{\frac{1}{2}} =: c_2 \cdot \theta^{\frac{1}{2}}. \end{aligned} \quad (5.7)$$

Ad. J_3^n . Let $\gamma > \frac{d}{2} + 1$. Similarly, since $B : H \times H \rightarrow V'_\gamma$ is bilinear and continuous, and the embedding $V'_\gamma \hookrightarrow U'$ is continuous, then by (5.4) we have the following estimates

$$\begin{aligned} \mathbb{E}[|J_3^n(\tau_n + \theta) - J_3^n(\tau_n)|_{U'}] &= \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n + \theta} B_n(u_n(s)) ds\right|_{U'}\right] \\ &\leq c \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} |B(u_n(s))|_{V'_\gamma} ds\right] \leq c \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|B\| \cdot |u_n(s)|_H^2 ds\right] \\ &\leq c \|B\| \cdot \mathbb{E}\left[\sup_{s \in [0, T]} |u_n(s)|_H^2\right] \cdot \theta \leq c \|B\| C_1(2) \cdot \theta =: c_3 \cdot \theta, \end{aligned} \quad (5.8)$$

where $\|B\|$ stands for the norm of $B : H \times H \rightarrow V'_\gamma$.

Ad. J_4^n . By the continuity of the embedding $U \hookrightarrow V$ we have

$$\begin{aligned} \mathbb{E}[|J_4^n(\tau_n + \theta) - J_4^n(\tau_n)|_{U'}] &= \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n + \theta} P_n f(s) ds\right|_{U'}\right] \\ &\leq c \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n + \theta} f(s) ds\right|_{V'}\right] \leq c\theta^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T |f(s)|_{V'}^2 ds\right]\right)^{\frac{1}{2}} \\ &= c\theta^{\frac{1}{2}} \|f\|_{L^2(0,T;V')} =: c_4 \cdot \theta^{\frac{1}{2}}. \end{aligned} \quad (5.9)$$

Ad. J_5^n . Since $V' \hookrightarrow U'$, then by (G*) and (5.4), we obtain the following inequalities

$$\begin{aligned} &\mathbb{E}[|J_5^n(\tau_n + \theta) - J_5^n(\tau_n)|_{U'}^2] \\ &= \mathbb{E}\left[\left|\int_{\tau_n}^{\tau_n + \theta} P_n G(u_n(s)) dW(s)\right|_{U'}^2\right] = \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|P_n G(u_n(s))\|_{\mathcal{H}_2(Y,U')}^2 ds\right] \\ &\leq c \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|G(u_n(s))\|_{\mathcal{H}_2(Y,V')}^2 ds\right] \leq cC \cdot \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} (1 + |u_n(s)|_H^2) ds\right] \\ &\leq cC(1 + \mathbb{E}[\sup_{s \in [0,T]} |u_n(s)|_H^2])\theta \leq cC(1 + C_1(2))\theta =: c_5 \cdot \theta. \end{aligned} \quad (5.10)$$

Let us fix $\eta > 0$ and $\varepsilon > 0$. By the Chebyshev inequality and estimates (5.7)-(5.9), we obtain for all $n \in \mathbb{N}$

$$\mathbb{P}(\{|J_i^n(\tau_n + \theta) - J_i^n(\tau_n)|_{U'} \geq \eta\}) \leq \frac{1}{\eta} \mathbb{E}[|J_i^n(\tau_n + \theta) - J_i^n(\tau_n)|_{U'}] \leq \frac{c_i \cdot \theta}{\eta},$$

where $i = 1, 2, 3, 4$. Let $\delta_i := \frac{\eta}{c_i} \cdot \varepsilon$. Then

$$\sup_{n \in \mathbb{N}} \sup_{1 \leq \theta \leq \delta_i} \mathbb{P}\{|J_i^n(\tau_n + \theta) - J_i^n(\tau_n)|_{U'} \geq \eta\} \leq \varepsilon, \quad i = 1, 2, 3, 4.$$

By the Chebyshev inequality and (5.10), we have for all $n \in \mathbb{N}$

$$\mathbb{P}(\{|J_5^n(\tau_n + \theta) - J_5^n(\tau_n)|_{U'} \geq \eta\}) \leq \frac{1}{\eta^2} \mathbb{E}[|J_5^n(\tau_n + \theta) - J_5^n(\tau_n)|_{U'}^2] \leq \frac{c_5 \cdot \theta}{\eta^2}.$$

Let $\delta_5 := \frac{\eta^2}{c_5} \cdot \varepsilon$. Then

$$\sup_{n \in \mathbb{N}} \sup_{1 \leq \theta \leq \delta_5} \mathbb{P}\{|J_5^n(\tau_n + \theta) - J_5^n(\tau_n)|_{U'} \geq \eta\} \leq \varepsilon.$$

Since condition **[A]** holds for each term J_i^n , $i = 1, 2, 3, 4, 5$, we infer that it holds also for (u_n) . This completes the proof of lemma. \square

5.3. Proof of Theorem 5.1. The following proof differs from the approach of Mikulevicius and Rozovskii [31] and it is based on the method used by Da Prato and Zabczyk in [16], Section 8, and on the Jakubowski's version of the Skorokhod Theorem for nonmetric spaces.

By Lemma 5.4 the set of measures $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$ is tight on the space $(\mathcal{Z}, \mathcal{T})$ defined by (3.14). Hence by Corollary 3.12 there exist a subsequence $(n_k)_k$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and, on this space, \mathcal{Z} -valued random variables $\tilde{u}, \tilde{u}_{n_k}, k \geq 1$ such that

$$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ on } \mathcal{Z} \text{ and } \tilde{u}_{n_k} \rightarrow \tilde{u} \text{ in } \mathcal{Z}, \quad \tilde{\mathbb{P}} - a.s. \quad (5.11)$$

Let us denote the subsequence $(\tilde{u}_{n_k})_k$ again by $(\tilde{u}_n)_n$.

Since $u_n \in \mathcal{C}([0, T]; P_n H)$, \mathbb{P} -a.s. and \tilde{u}_n and u_n have the same laws, and $\mathcal{C}([0, T]; P_n H)$ is a Borel subset of $\mathcal{C}([0, T]; U') \cap L^2(0, T; H_{loc})$, we have

$$\mathcal{L}(\tilde{u}_n)(\mathcal{C}([0, T]; P_n H)) = 1, \quad n \geq 1.$$

Since \tilde{u}_n and u_n have the same laws, and $\mathcal{C}([0, T]; P_n H)$ is a Borel subset of $\mathcal{C}([0, T]; U') \cap L^2(0, T; H_{loc})$ thus by (5.4) and (5.6) we have

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left(\sup_{0 \leq s \leq T} |\tilde{u}_n(s)|_H^p \right) \leq C_1(p), \quad (5.12)$$

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\int_0^T \|\tilde{u}_n(s)\|_V^2 ds \right] \leq C_2 \quad (5.13)$$

for all p satisfying condition (5.3).

By inequality (5.13) we infer that the sequence (\tilde{u}_n) contain subsequence, still denoted by (\tilde{u}_n) convergent weakly in the space $L^2([0, T] \times \tilde{\Omega}; V)$. Since by (5.11) $\tilde{\mathbb{P}}$ -a.s. $\tilde{u}_n \rightarrow \tilde{u}$ in \mathcal{Z} , we conclude that $\tilde{u} \in L^2([0, T] \times \tilde{\Omega}; V)$, i.e.

$$\mathbb{E} \left[\int_0^T \|\tilde{u}(s)\|^2 ds \right] < \infty. \quad (5.14)$$

Similarly, by inequality (5.12) with $p = 2$ we can choose a subsequence of (\tilde{u}_n) convergent weak star in the space $L^2(\tilde{\Omega}; L^\infty(0, T; H))$ and, using (5.11), infer that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |\tilde{u}(s)|_H^2 \right] < \infty. \quad (5.15)$$

For each $n \geq 1$, let us consider a process \tilde{M}_n with trajectories in $\mathcal{C}([0, T]; H)$ defined by

$$\tilde{M}_n(t) = \tilde{u}_n(t) - P_n \tilde{u}(0) + \int_0^t P_n \mathcal{A} \tilde{u}_n(s) ds + \int_0^t B_n(\tilde{u}_n(s)) ds - \int_0^t P_n f(s) ds, \quad (5.16)$$

where $t \in [0, T]$. \tilde{M}_n is a square integrable martingale with respect to the filtration $\tilde{\mathbb{F}}_n = (\tilde{\mathcal{F}}_{n,t})$, where $\tilde{\mathcal{F}}_{n,t} = \sigma\{\tilde{u}_n(s), s \leq t\}$, with quadratic variation

$$\langle\langle \tilde{M}_n \rangle\rangle_t = \int_0^t P_n G(\tilde{u}_n(s)) G(\tilde{u}_n(s))^* P_n ds, \quad t \in [0, T]. \quad (5.17)$$

Indeed, since \tilde{u}_n and u_n have the same laws, for all $s, t \in [0, T]$, $s \leq t$ all functions h bounded continuous on $\mathcal{C}([0, s]; U')$, and all $\psi, \zeta \in U$, we have

$$\mathbb{E}[\langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle h(\tilde{u}_{n|[0,s]})] = 0 \quad (5.18)$$

and

$$\begin{aligned} & \mathbb{E} \left[\left(\langle \tilde{M}_n(t), \psi \rangle \langle \tilde{M}_n(t), \zeta \rangle - \langle \tilde{M}_n(s), \psi \rangle \langle \tilde{M}_n(s), \zeta \rangle \right. \right. \\ & \quad \left. \left. - \int_s^t \left(G(\tilde{u}_n(\sigma))^* P_n \psi, G(\tilde{u}_n(\sigma))^* P_n \zeta \right)_Y d\sigma \right) \cdot h(\tilde{u}_{n|[0,s]}) \right] = 0 \end{aligned} \quad (5.19)$$

Here, $\langle \cdot, \cdot \rangle$ stands for the dual pairing between U' and U . Let us recall that Y is a Hilbert space defined in assumption A.1. We will take the limits in (5.18) and (5.19). Let \tilde{M} be an U' -valued process defined by

$$\tilde{M}(t) = \tilde{u}(t) - \tilde{u}(0) + \int_0^t \mathcal{A}\tilde{u}(s) ds + \int_0^t B(\tilde{u}(s)) ds - \int_0^t f(s) ds, \quad t \in [0, T]. \quad (5.20)$$

Lemma 5.5. *For all $s, t \in [0, T]$ such that $s \leq t$ and all $\psi \in U$:*

- (a) $\lim_{n \rightarrow \infty} (\tilde{u}_n(t), P_n \psi)_H = (\tilde{u}(t), \psi)_H, \quad \tilde{\mathbb{P}} - a.s.,$
- (b) $\lim_{n \rightarrow \infty} \int_s^t \langle \mathcal{A}\tilde{u}_n(\sigma), P_n \psi \rangle d\sigma = \int_s^t \langle \mathcal{A}\tilde{u}(\sigma), \psi \rangle d\sigma, \quad \tilde{\mathbb{P}} - a.s.,$
- (c) $\lim_{n \rightarrow \infty} \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n \psi \rangle d\sigma = \int_s^t \langle B(\tilde{u}(\sigma)), \psi \rangle d\sigma, \quad \tilde{\mathbb{P}} - a.s.$

Proof. Let us fix $s, t \in [0, T]$, $s \leq t$ and $\psi \in U$. By (5.11) we know that

$$\tilde{u}_n \rightarrow \tilde{u} \text{ in } \mathcal{C}([0, T]; U') \cap L_w^2(0, T; V) \cap L^2(0, T; H_{loc}) \cap \mathcal{C}([0, T]; H_w), \quad \tilde{\mathbb{P}} - a.s. \quad (5.21)$$

Thus $\tilde{u}_n \rightarrow \tilde{u}$ in $\mathcal{C}([0, T], H_w)$ - $\tilde{\mathbb{P}}$ a.s. and since by (2.26) $P_n \psi \rightarrow \psi$ in H , we infer that assertion (a) holds.

Let us move to (b). Since by (5.21) $\tilde{u}_n \rightarrow \tilde{u}$ in $L_w^2(0, T; V)$ - $\tilde{\mathbb{P}}$ - a.s. and by assertion (iii) in Lemma 2.4 (c) $P_n \psi \rightarrow \psi$ in V , by (2.15) we infer that $\tilde{\mathbb{P}}$ - a.s.

$$\begin{aligned} \int_s^t \langle \mathcal{A}\tilde{u}_n(\sigma), P_n \psi \rangle d\sigma &= \int_s^t ((\tilde{u}_n(\sigma), P_n \psi)) d\sigma \\ &\xrightarrow{n \rightarrow \infty} \int_s^t ((\tilde{u}(\sigma), \psi)) d\sigma = \int_s^t \langle \mathcal{A}\tilde{u}(\sigma), \psi \rangle d\sigma, \end{aligned}$$

i.e. (b) holds.

We will prove now assertion (c). Since as above $\tilde{u}_n(\cdot, \omega) \rightarrow \tilde{u}(\cdot, \omega)$ in $L_w^2(0, T; V)$, in particular, $\tilde{u}(\cdot, \omega) \in L^2(0, T; V)$ and the sequence $(\tilde{u}_n(\cdot, \omega))_{n \geq 1}$ is bounded in $L^2(0, T; V)$ for $\tilde{\mathbb{P}}$ -almost all $\omega \in \tilde{\Omega}$. Thus $\tilde{u}(\cdot, \omega) \in L^2(0, T; H)$ and the sequence $(\tilde{u}_n(\cdot, \omega))_{n \geq 1}$ is bounded in $L^2(0, T; H)$, as well. Let us fix $\omega \in \tilde{\Omega}$ such that

- (i) $\tilde{u}_n(\cdot, \omega) \rightarrow \tilde{u}(\cdot, \omega)$ in $L^2(0, T, H_{loc}) \cap \mathcal{C}([0, T]; U')$,
- (ii) $\tilde{u}(\cdot, \omega) \in L^2(0, T; H)$ and the sequence $(\tilde{u}_n(\cdot, \omega))_{n \geq 1}$ is bounded in $L^2(0, T; H)$.

By (i) the sequence $(\tilde{u}_n(\cdot, \omega))_{n \geq 1}$ is bounded in $\mathcal{C}([0, T]; U')$, i.e. for some $N > 0$

$$\sup_{n \geq 1} \|\tilde{u}_n(\cdot, \omega)\|_{\mathcal{C}([0, T]; U')} \leq N.$$

Thus $\chi_n(\tilde{u}_n(\cdot, \omega)) = \tilde{u}_n(\cdot, \omega)$ for all $n > N$ and

$$B(\chi_n(\tilde{u}_n(\cdot, \omega)), \tilde{u}_n(\cdot, \omega)) = B(\tilde{u}_n(\cdot, \omega), \tilde{u}_n(\cdot, \omega)) \quad \text{for } n > N.$$

Hence assertion (c) follows from Corollary B.2. This completes the proof of the lemma. \square

Lemma 5.6. *For all $s, t \in [0, T]$ such that $s \leq t$ and all $\psi \in U$:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle h(\tilde{u}_{n|[0, s]})] = \mathbb{E}[\langle \tilde{M}(t) - \tilde{M}(s), \psi \rangle h(\tilde{u}_{|[0, s]})].$$

Proof. Let us fix $s, t \in [0, T]$, $s \leq t$ and $\psi \in U$. By (2.27) we have

$$\begin{aligned} \langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle &= (\tilde{u}_n(t), P_n \psi)_H - (\tilde{u}_n(s), P_n \psi)_H + \int_s^t \langle \mathcal{A} \tilde{u}_n(\sigma), P_n \psi \rangle d\sigma \\ &+ \int_s^t \langle B(\chi_n(\tilde{u}_n(\sigma)), \tilde{u}_n(\sigma)), P_n \psi \rangle d\sigma + \int_s^t \langle f(\sigma), P_n \psi \rangle d\sigma. \end{aligned}$$

By Lemma 5.5, we infer that

$$\lim_{n \rightarrow \infty} \langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle = \langle \tilde{M}(t) - \tilde{M}(s), \psi \rangle, \quad \tilde{\mathbb{P}} \text{ - a.s.} \quad (5.22)$$

Let us notice that $\tilde{\mathbb{P}}$ - a.s. $\lim_{n \rightarrow \infty} h(\tilde{u}_{n|[0, s]}) = h(\tilde{u}_{|[0, s]})$ and $\sup_{n \in \mathbb{N}} \|h(\tilde{u}_{n|[0, s]})\|_{L^\infty} < \infty$. Let us denote

$$f_n(\omega) := (\langle \tilde{M}_n(t, \omega), \psi \rangle - \langle \tilde{M}_n(s, \omega), \psi \rangle) h(\tilde{u}_{n|[0, s]}), \quad \omega \in \tilde{\Omega}.$$

We will prove that the functions $\{f_n\}_{n \in \mathbb{N}}$ are uniformly integrable. We claim that

$$\sup_{n \geq 1} \tilde{\mathbb{E}}[|f_n|^2] < \infty. \quad (5.23)$$

Indeed, by the continuity of the embedding $U \hookrightarrow H$ and the Schwarz inequality, for each $n \in \mathbb{N}$ we have

$$\tilde{\mathbb{E}}[|f_n|^2] \leq 2c \|h\|_{L^\infty}^2 \|\psi\|_{\tilde{U}}^2 \tilde{\mathbb{E}}[|\tilde{M}_n(t)|_H^2 + |\tilde{M}_n(s)|_H^2]. \quad (5.24)$$

Since \tilde{M}_n is a continuous martingale with quadratic variation defined in (5.17), by the Burkholder-Davis-Gundy inequality we obtain

$$\tilde{\mathbb{E}}\left[\sup_{t \in [0, T]} |\tilde{M}_n(t)|^2\right] \leq c \tilde{\mathbb{E}}\left[\left(\int_0^T \|P_n G(\tilde{u}_n(\sigma))\|_{\mathcal{T}_2(Y, H)}^2 d\sigma\right)^{\frac{1}{2}}\right]. \quad (5.25)$$

Since the restriction of P_n to H is an orthogonal projection onto H_n , by inequality (G) in assumption (A.3), we have for all $\sigma \in [0, T]$

$$\|P_n G(\tilde{u}_n(\sigma))\|_{\mathcal{T}_2(Y, H)}^2 \leq (2 - \eta) \|\tilde{u}_n(\sigma)\|^2 + \lambda_0 |\tilde{u}_n(\sigma)|_H^2 + \rho. \quad (5.26)$$

By (5.25), (5.26), (5.13) and (5.12), we infer that

$$\sup_{n \in \mathbb{N}} \tilde{\mathbb{E}}\left[\sup_{t \in [0, T]} |\tilde{M}_n(t)|^2\right] < \infty. \quad (5.27)$$

Then by (5.24) and (5.27) we see that (5.23) holds. Since the sequence $\{f_n\}_{n \in \mathbb{N}}$ is uniformly integrable and by (5.22) it is $\tilde{\mathbb{P}}$ -a.s. pointwise convergent, application of the Vitali Theorem completes the proof of the Lemma. \square

Lemma 5.7. *For all $s, t \in [0, T]$ such that $s \leq t$ and all $\psi, \zeta \in U$:*

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}\left[\left\{\langle \tilde{M}_n(t), \psi \rangle \langle \tilde{M}_n(t), \zeta \rangle - \langle \tilde{M}_n(s), \psi \rangle \langle \tilde{M}_n(s), \zeta \rangle\right\} h(\tilde{u}_{n|_{[0, s]}})\right] \\ = \mathbb{E}\left[\left\{\langle \tilde{M}(t), \psi \rangle \langle \tilde{M}(t), \zeta \rangle - \langle \tilde{M}(s), \psi \rangle \langle \tilde{M}(s), \zeta \rangle\right\} h(\tilde{u}_{|[0, s]})\right]. \end{aligned}$$

Proof. Let us fix $s, t \in [0, T]$ such that $s \leq t$ and $\psi, \zeta \in U$ and let us denote

$$\begin{aligned} f_n(\omega) &:= \left\{\langle \tilde{M}_n(t, \omega), \psi \rangle \langle \tilde{M}_n(t, \omega), \zeta \rangle - \langle \tilde{M}_n(s, \omega), \psi \rangle \langle \tilde{M}_n(s, \omega), \zeta \rangle\right\} \\ &\quad \cdot h(\tilde{u}_{n|_{[0, s]}}(\omega)), \\ f(\omega) &:= \left\{\langle \tilde{M}(t, \omega), \psi \rangle \langle \tilde{M}(t, \omega), \zeta \rangle - \langle \tilde{M}(s, \omega), \psi \rangle \langle \tilde{M}(s, \omega), \zeta \rangle\right\} \\ &\quad \cdot h(\tilde{u}_{|[0, s]}(\omega)), \quad \omega \in \tilde{\Omega}. \end{aligned}$$

By Lemma 5.5, we infer that $\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$ for $\tilde{\mathbb{P}}$ -almost all $\omega \in \tilde{\Omega}$.

We will prove that the functions $\{f_n\}_{n \in \mathbb{N}}$ are uniformly integrable. To this end, it is sufficient to show that for some $r > 1$,

$$\sup_{n \geq 1} \mathbb{E}[|f_n|^r] < \infty. \quad (5.28)$$

In fact, we will show that condition (5.28) holds for any $r \in (1, 1 + \frac{\eta}{2(2-\eta)})$ if $0 < \eta < 2$ and any $r > 1$ if $\eta = 2$. Indeed, for each $n \in \mathbb{N}$ we have

$$\mathbb{E}[|f_n|^r] \leq C \|h\|_{L^\infty}^r \|\psi\|_U^r \|\eta\|_U^r \mathbb{E}[|\tilde{M}_n(t)|^{2r} + |\tilde{M}_n(s)|^{2r}]. \quad (5.29)$$

Since \tilde{M}_n is a continuous martingale with quadratic variation defined in (5.17), by the Burkholder-Davis-Gundy inequality we obtain

$$\mathbb{E}\left[\sup_{t \in [0, T]} |\tilde{M}_n(t)|^{2r}\right] \leq c\mathbb{E}\left[\left(\int_0^T \|P_n G(\tilde{u}_n(\sigma))\|_{\mathcal{H}_2(Y, U')}^2 d\sigma\right)^r\right] \quad (5.30)$$

By the continuity of the injection $U \hookrightarrow V$, lemma 2.4 and assumption (G*) we have for all $\sigma \in [0, T]$

$$\|P_n G(\tilde{u}_n(\sigma))\|_{\mathcal{H}_2(Y, U')}^2 \leq c|P_n|_{\mathcal{L}(U, V)}\|G(\tilde{u}_n(\sigma))\|_{\mathcal{H}_2(Y, V')}^2 \leq C(|\tilde{u}_n(\sigma)|_H^2 + 1).$$

Since $2r$ satisfies condition (5.3), by (5.12) we obtain the following inequalities

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T \|P_n G(\tilde{u}_n(s))\|_{\mathcal{H}_2(Y, U')}^2 ds\right)^r\right] &\leq \tilde{C}\mathbb{E}\left[\left(\int_0^T (|\tilde{u}_n(s)|_H^2 + 1) ds\right)^r\right] \\ &\leq c\left(\mathbb{E}\left[\sup_{s \in [0, T]} |\tilde{u}_n(s)|_H^{2r}\right] + 1\right) \leq c(C_1(2r) + 1). \end{aligned} \quad (5.31)$$

By (5.30), (5.31) and (5.29) we infer that condition (5.28) holds. By the Vitali Theorem

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}}[f_n] = \tilde{\mathbb{E}}[f].$$

The proof of the Lemma is thus complete. \square

Lemma 5.8. (Convergence in quadratic variation). For any $s, t \in [0, T]$ and $\psi, \zeta \in U$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}\left[\left(\int_s^t (G(\tilde{u}_n(\sigma))^* P_n \psi, G(\tilde{u}_n(\sigma))^* P_n \zeta)_Y d\sigma\right) \cdot h(\tilde{u}_n|_{[0, s]})\right] \\ = \mathbb{E}\left[\left(\int_s^t (G(\tilde{u}(\sigma))^* \psi, G(\tilde{u}(\sigma))^* \zeta)_Y d\sigma\right) \cdot h(\tilde{u}|_{[0, s]})\right]. \end{aligned}$$

Let us recall that Y is the Hilbert space defined in assumption A.1 in Section 4.

Proof. Let us fix $\psi, \zeta \in U$ and let us denote

$$f_n(\omega) := \left(\int_s^t (G(\tilde{u}_n(\sigma, \omega))^* P_n \psi, G(\tilde{u}_n(\sigma, \omega))^* P_n \zeta)_Y d\sigma\right) \cdot h(\tilde{u}_n|_{[0, s]}), \quad \omega \in \tilde{\Omega}.$$

We will prove that the functions are uniformly integrable and convergent $\tilde{\mathbb{P}}$ -a.s.

Uniform integrability. It is sufficient to show that for some $r > 1$

$$\sup_{n \geq 1} \mathbb{E}[|f_n|^r] < \infty. \quad (5.32)$$

We will prove that the above condition holds for every $r \in (1, 1 + \frac{\eta}{2(2-\eta)})$ if $0 < \eta < 2$ and with every $r > 1$ if $\eta = 2$.

Since $\mathcal{T}_2(Y, V') \hookrightarrow \mathcal{L}(Y, V')$ and $U \hookrightarrow V$ continuously, then by (G^*) and Lemma 2.4 we have the following inequalities

$$\begin{aligned} \|G(\tilde{u}_n(\sigma, \omega))^* P_n \zeta\|_Y &\leq \|G(\tilde{u}_n(\sigma, \omega))\|_{\mathcal{L}(Y, V')} \cdot \|P_n \zeta\|_V \\ &\leq c \sqrt{(|\tilde{u}_n(\sigma, \omega)|_H^2 + 1)} \|\zeta\|_U \end{aligned}$$

for some $c > 0$. Thus we have the following inequalities

$$\begin{aligned} |f_n|^r &= \left| \left(\int_s^t \left(G(\tilde{u}_n(\sigma, \omega))^* P_n \psi, G(\tilde{u}_n(\sigma, \omega))^* P_n \zeta \right)_Y d\sigma \right) \cdot h(\tilde{u}_n|_{[0, s]}) \right|^r \\ &\leq \|h\|_{L^\infty}^r \left(\int_s^t \|G(\tilde{u}_n(\sigma, \omega))^* P_n \psi\|_Y \cdot \|G(\tilde{u}_n(\sigma, \omega))^* P_n \zeta\|_Y d\sigma \right)^r \\ &\leq c^{2r} \|h\|_{L^\infty}^r \cdot \|\psi\|_U^r \cdot \|\zeta\|_U^r \cdot \left(\int_s^t (|\tilde{u}_n(\sigma, \omega)|_H^2 + 1) d\sigma \right)^r. \end{aligned}$$

Using the Hölder inequality, we obtain for some $C > 0$

$$\begin{aligned} \left(\int_s^t (|\tilde{u}_n(\sigma, \omega)|_H^2 + 1) d\sigma \right)^r &\leq (t-s)^{r-1} \cdot \int_s^t (|\tilde{u}_n(\sigma, \omega)|_H^2 + 1)^r d\sigma \\ &\leq C \cdot \sup_{\sigma \in [0, T]} (|\tilde{u}_n(\sigma, \omega)|_H^{2r} + 1). \end{aligned}$$

Thus

$$|f_n|^r \leq \tilde{C} \cdot \sup_{\sigma \in [0, T]} (|\tilde{u}_n(\sigma, \omega)|_H^{2r} + 1)$$

for some $\tilde{C} > 0$. Hence by (5.12)

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|f_n|^r] \leq \tilde{C} \cdot \mathbb{E} \left[\sup_{\sigma \in [0, T]} |\tilde{u}_n(\sigma, \omega)|_H^{2r} + 1 \right] \leq \tilde{C} (C_1(2r) + 1) < \infty.$$

Thus condition (5.32) holds.

Pointwise convergence on $\tilde{\Omega}$. Let us fix $\omega \in \tilde{\Omega}$ such that

- (i) $\tilde{u}_n(\cdot, \omega) \rightarrow \tilde{u}(\cdot, \omega)$ in $L^2(0, T, H_{loc})$,
- (ii) $\tilde{u}(\cdot, \omega) \in L^2(0, T; H)$ and the sequence $(\tilde{u}_n(\cdot, \omega))_{n \geq 1}$ is bounded in $L^2(0, T; H)$.

We will prove that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_s^t \left(G(\tilde{u}_n(\sigma, \omega))^* P_n \psi, G(\tilde{u}_n(\sigma, \omega))^* P_n \zeta \right)_Y d\sigma \\ = \int_s^t \left(G(\tilde{u}(\sigma, \omega))^* \psi, G(\tilde{u}(\sigma, \omega))^* \zeta \right)_Y d\sigma. \end{aligned}$$

Let us notice that it is sufficient to prove that

$$G(\tilde{u}_n(\cdot, \omega))^* P_n \psi \rightarrow G(\tilde{u}(\cdot, \omega))^* \psi \quad \text{in } L^2(s, t; Y). \quad (5.33)$$

We have

$$\begin{aligned}
& \int_s^t \|G(\tilde{u}_n(\sigma, \omega))^* P_n \psi - G(\tilde{u}(\sigma, \omega))^* \psi\|_Y^2 d\sigma \\
& \leq \int_s^t \left(\|G(\tilde{u}_n(\sigma, \omega))^* (P_n \psi - \psi)\|_Y + \|G(\tilde{u}_n(\sigma, \omega))^* \psi - G(\tilde{u}(\sigma, \omega))^* \psi\|_Y \right)^2 d\sigma \\
& \leq 2 \int_s^t |G(\tilde{u}_n(\sigma, \omega))^*|_{\mathcal{L}(V', Y)}^2 \cdot \|P_n \psi - \psi\|_V^2 d\sigma \\
& + 2 \int_s^t \|G(\tilde{u}_n(\sigma, \omega))^* \psi - G(\tilde{u}(\sigma, \omega))^* \psi\|_Y^2 d\sigma \\
& =: 2\{I_1(n) + I_2(n)\}.
\end{aligned} \tag{5.34}$$

Let us consider the term $I_1(n)$. Since $\psi \in U$, by assertion (iii) in Lemma 2.4 (c), we have

$$\lim_{n \rightarrow \infty} \|P_n \psi - \psi\|_V = 0.$$

By (G*), the continuity of the embedding $\mathcal{T}_2(Y, V') \hookrightarrow \mathcal{L}(Y, V')$ and (ii), we infer that

$$\begin{aligned}
& \int_s^t |G(\tilde{u}_n(\sigma, \omega))^*|_{\mathcal{L}(V', Y)}^2 d\sigma \leq C \int_s^t (|\tilde{u}_n(\sigma, \omega)|_H^2 + 1) d\sigma \\
& \leq \tilde{C} \left(\sup_{n \geq 1} \|\tilde{u}_n(\cdot, \omega)\|_{L^2(0, T; H)}^2 + 1 \right) \leq K
\end{aligned}$$

for some constant $K > 0$. Thus

$$\lim_{n \rightarrow \infty} I_1(n) = \lim_{n \rightarrow \infty} \int_s^t |G(\tilde{u}_n(\sigma, \omega))^*|_{\mathcal{L}(H, Y)}^2 \cdot \|P_n \psi - \psi\|_V^2 d\sigma = 0.$$

Let us move to the term $I_2(n)$ in (5.34). We will prove that for every $\psi \in V$ the term $I_2(n)$ tends to zero as $n \rightarrow \infty$. Assume first that $\psi \in \mathcal{V}$. Then there exists $R > 0$ such that $\text{supp } \psi$ is a compact subset of \mathcal{O}_R . Since $\tilde{u}_n(\cdot, \omega) \rightarrow \tilde{u}(\cdot, \omega)$ in $L^2(0, T; H_{loc})$, then in particular

$$\lim_{n \rightarrow \infty} q_{T, R}(\tilde{u}_n(\cdot, \omega) - \tilde{u}(\cdot, \omega)) = 0,$$

where $q_{T, R}$ is the seminorm defined by (3.6). In the other words, $\tilde{u}_n(\cdot, \omega) \rightarrow \tilde{u}(\cdot, \omega)$ in $L^2(0, T; H_{\mathcal{O}_R})$. Therefore there exists a subsequence $(\tilde{u}_{n_k}(\cdot, \omega))_k$ such that

$$\tilde{u}_{n_k}(\sigma, \omega) \rightarrow \tilde{u}(\sigma, \omega) \quad \text{in } H_{\mathcal{O}_R} \text{ for almost all } \sigma \in [0, T] \text{ as } k \rightarrow \infty.$$

Hence by assumption (G**)

$$G(\tilde{u}_{n_k}(\sigma, \omega))^* \psi \rightarrow G(\tilde{u}(\sigma, \omega))^* \psi \quad \text{in } Y \text{ for almost all } \sigma \in [0, T] \text{ as } k \rightarrow \infty.$$

In conclusion, by the Vitali Theorem

$$\lim_{k \rightarrow \infty} \int_s^t \|G(\tilde{u}_{n_k}(\sigma, \omega))^* \psi - G(\tilde{u}(\sigma, \omega))^* \psi\|_Y^2 d\sigma = 0 \quad \text{for } \psi \in \mathcal{V}.$$

Repeating the above reasoning for all subsequences, we infer that from every subsequence of the sequence $(G(\tilde{u}_n(\sigma, \omega))^* \psi)_n$ we can choose the subsequence convergent in $L^2(s, t; Y)$ to the same limit. Thus the whole sequence $(G(\tilde{u}_n(\sigma, \omega))^* \psi)_n$ is convergent to $G(\tilde{u}(\sigma, \omega))^* \psi$ in $L^2(s, t; Y)$. At the same time

$$\lim_{n \rightarrow \infty} I_2(n) = 0 \quad \text{for every } \psi \in \mathcal{V}.$$

If $\psi \in V$ then for every $\varepsilon > 0$ we can find $\psi_\varepsilon \in \mathcal{V}$ such that $\|\psi - \psi_\varepsilon\|_V < \varepsilon$. By the continuity of the embedding $\mathcal{T}_2(Y, V') \hookrightarrow \mathcal{L}(Y, V')$, condition (G^*) and (ii), we obtain

$$\begin{aligned} & \int_s^t \|G(\tilde{u}_n(\sigma, \omega))^* \psi - G(\tilde{u}(\sigma, \omega))^* \psi\|_Y^2 d\sigma \\ & \leq 2 \int_s^t \| [G(\tilde{u}_n(\sigma, \omega))^* - G(\tilde{u}(\sigma, \omega))^*] (\psi - \psi_\varepsilon) \|_Y^2 d\sigma \\ & \quad + 2 \int_s^t \| [G(\tilde{u}_n(\sigma, \omega))^* - G(\tilde{u}(\sigma, \omega))^*] \psi_\varepsilon \|_Y^2 d\sigma \\ & \leq 4 \int_s^t [\|G(\tilde{u}_n(\sigma, \omega))\|_{\mathcal{L}(Y, V')}^2 + \|G(\tilde{u}(\sigma, \omega))\|_{\mathcal{L}(Y, V')}^2] \|\psi - \psi_\varepsilon\|_V^2 d\sigma \\ & \quad + 2 \int_s^t \| [G(\tilde{u}_n(\sigma, \omega))^* - G(\tilde{u}(\sigma, \omega))^*] \psi_\varepsilon \|_Y^2 d\sigma \\ & \leq c(\|\tilde{u}_n(\cdot, \omega)\|_{L^2(0, T; H)}^2 + \|\tilde{u}(\cdot, \omega)\|_{L^2(0, T; H)}^2 + 2T) \cdot \varepsilon^2 \\ & \quad + 2 \int_s^t \| [G(\tilde{u}_n(\sigma, \omega))^* - G(\tilde{u}(\sigma, \omega))^*] \psi_\varepsilon \|_Y^2 d\sigma \\ & \leq C\varepsilon^2 + 2 \int_s^t \| [G(\tilde{u}_n(\sigma, \omega))^* - G(\tilde{u}(\sigma, \omega))^*] \psi_\varepsilon \|_Y^2 d\sigma, \end{aligned}$$

for some positive constants c and C . Passing to the upper limit as $n \rightarrow \infty$, we infer that

$$\limsup_{n \rightarrow \infty} \int_s^t \|G(\tilde{u}_n(\sigma, \omega))^* \psi - G(\tilde{u}(\sigma, \omega))^* \psi\|_Y^2 d\sigma \leq C\varepsilon^2.$$

In conclusion, we proved that

$$\lim_{n \rightarrow \infty} \int_s^t \|G(\tilde{u}_n(\sigma, \omega))^* \psi - G(\tilde{u}(\sigma, \omega))^* \psi\|_Y^2 d\sigma = 0$$

which completes the proof of (5.33) and of Lemma 5.8. \square

By Lemma 5.6 we can pass to the limit in (5.18). By Lemmas 5.7 and 5.8 we can pass to the limit in (5.19) as well. After passing to the limits in (5.18) and (5.19) we infer that for all $\psi, \zeta \in U$:

$$\mathbb{E}[\langle \tilde{M}(t) - \tilde{M}(s), \psi \rangle h(\tilde{u}|_{[0, s]})] = 0 \quad (5.35)$$

and

$$\mathbb{E} \left[\left(\langle \tilde{M}(t), \psi \rangle \langle \tilde{M}(t), \zeta \rangle - \langle \tilde{M}(s), \psi \rangle \langle \tilde{M}(s), \zeta \rangle - \int_s^t (G(\tilde{u}(\sigma))^* \psi, G(\tilde{u}(\sigma))^* \zeta)_Y d\sigma \right) \cdot h(\tilde{u}|_{[0,s]}) \right] = 0. \quad (5.36)$$

where \tilde{M} is an U' -valued process defined by (5.20).

Continuation of the proof of Theorem 5.1. Now, we apply the idea analogous to the reasoning used by Da Prato and Zabczyk, see [16], Section 8.3. Consider operator $L : U \supset D(L) \rightarrow H$ defined by (2.19), the inverse $L^{-1} : H \rightarrow U$ and its dual $(L^{-1})' : U' \rightarrow H'$. By (5.35) and (5.36) with $\psi := L^{-1}\varphi$ and $\zeta := L^{-1}\eta$, where $\varphi, \eta \in H$ and equality (2.23), we infer that

$(L^{-1})'\tilde{M}(t)$, $t \in [0, T]$ is a continuous square integrable martingale in $H' \cong H$ with respect to the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)$,
where $\mathcal{F}_t = \sigma\{\tilde{u}(s), s \leq t\}$ with the quadratic variation

$$\langle\langle (L^{-1})'\tilde{M} \rangle\rangle_t = \int_0^t (L^{-1})'G(\tilde{u}(s))(G(\tilde{u}(s))(L^{-1})')^* ds.$$

In particular, the continuity of the process $(L^{-1})'\tilde{M}$ follows from the fact that $\tilde{u} \in \mathcal{C}([0, T]; U')$. By the martingale representation Theorem, see [16],

there exist

- a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}})$,
- a cylindrical Wiener process $\tilde{W}(t)$ defined on this basis,
- and a progressively measurable process $\tilde{u}(t)$ such that

$$\begin{aligned} (L^{-1})'\tilde{u}(t) - (L^{-1})'\tilde{u}(0) &+ (L^{-1})' \int_0^t \mathcal{A}\tilde{u}(s) ds + (L^{-1})' \int_0^t B(\tilde{u}(s), \tilde{u}(s)) ds \\ &- (L^{-1})' \int_0^t f(s) ds = \int_0^t (L^{-1})'G(\tilde{u}(s)) d\tilde{W}(s). \end{aligned}$$

However,

$$\int_0^t (L^{-1})'G(\tilde{u}(s)) d\tilde{W}(s) = (L^{-1})' \int_0^t G(\tilde{u}(s)) d\tilde{W}(s).$$

Hence for all $t \in [0, T]$ and all $v \in U$

$$\begin{aligned} (\tilde{u}(t), v)_H - (\tilde{u}(0), v)_H &+ \int_0^t \langle \mathcal{A}\tilde{u}(s), v \rangle ds + \int_0^t \langle B(\tilde{u}(s)), v \rangle ds \\ &= \int_0^t \langle f(s), v \rangle ds + \left\langle \int_0^t G(\tilde{u}(s)) d\tilde{W}(s), v \right\rangle. \end{aligned}$$

Thus the conditions from Definition 4.1 hold with $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}}) = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}})$, $\hat{W} = \tilde{W}$ and $u = \tilde{u}$. The proof of Theorem 5.1 is thus complete. \square

6. AN EXAMPLE

Let

$$G(u)(t, x) dW(t) := \sum_{i=1}^{\infty} [(b^{(i)}(x) \cdot \nabla) u(t, x) + c^{(i)}(x) u(t, x)] d\beta^{(i)}(t), \quad (6.1)$$

where

$$\begin{aligned} &\beta^{(i)}, i \in \mathbb{N} \text{ - independent standard Brownian motions,} \\ &b^{(i)} : \overline{\mathcal{O}} \rightarrow \mathbb{R}^d \text{ - of class } \mathcal{C}^\infty, \quad i \in \mathbb{N} \\ &c^{(i)} : \overline{\mathcal{O}} \rightarrow \mathbb{R} \text{ - of class } \mathcal{C}^\infty, \quad i \in \mathbb{N} \end{aligned}$$

are given. Assume that

$$C_1 := \sum_{i=1}^{\infty} (\|b^{(i)}\|_{L^\infty}^2 + \|\operatorname{div} b^{(i)}\|_{L^\infty}^2 + \|c^{(i)}\|_{L^\infty}^2) < \infty \quad (6.2)$$

and

$$\sum_{j,k=1}^d (2\delta_{jk} - \sum_{i=1}^{\infty} b_j^{(i)}(x) b_k^{(i)}(x)) \zeta_j \zeta_k \geq a|\zeta|^2, \quad \zeta \in \mathbb{R}^d \quad (6.3)$$

for some $a \in (0, 2]$. Assumption (6.3) is equivalent to the following one

$$\sum_{i=1}^{\infty} \sum_{j,k=1}^d b_j^{(i)}(x) b_k^{(i)}(x) \zeta_j \zeta_k \leq 2 \sum_{j,k=1}^d \delta_{jk} \zeta_j \zeta_k - a|\zeta|^2 = (2-a)|\zeta|^2. \quad (6.4)$$

Let $Y := l^2(\mathbb{N})$, where $l^2(\mathbb{N})$ denotes the space of all sequences $(h_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ such that $\sum_{i=1}^{\infty} h_i^2 < \infty$. It is a Hilbert space with the scalar product given by $(h, k)_{l^2} := \sum_{i=1}^{\infty} h_i k_i$, where $h = (h_i)$ and $k = (k_i)$ belong to $l^2(\mathbb{N})$. Let us put

$$G(u)h = \sum_{i=1}^{\infty} [(b^{(i)} \cdot \nabla) u + c^{(i)} u] h_i, \quad u \in V, \quad h = (h_i) \in l^2(\mathbb{N}). \quad (6.5)$$

We will show that the mapping G fulfils assumption **(A.3)**. Since G is linear, it is Lipschitz continuous provided that it is bounded. Thus we will show that

- The following inequality holds

$$2\langle \mathcal{A}u, u \rangle - \|G(u)\|_{\mathcal{T}_2(Y, H)}^2 \geq \eta \|u\|^2 - \lambda_0 |u|_H^2, \quad u \in V \quad (\tilde{\mathbf{G}})$$

for some constants λ_0 and $\eta \in (0, 2]$.

- Moreover, G extends to a linear mapping $G : H \rightarrow \mathcal{T}_2(Y, V')$ and

$$\|G(u)\|_{\mathcal{T}_2(Y, V')} \leq C|u|_H, \quad u \in H. \quad (\tilde{\mathbf{G}}^*)$$

for some $C > 0$.

- Furthermore, for each $R > 0$ the mapping $G : H_{\mathcal{O}_R} \rightarrow \mathcal{T}_2(Y, V'(\mathcal{O}_R))$ is well-defined and satisfies the following estimate

$$\|G(u)\|_{\mathcal{T}_2(Y, V'(\mathcal{O}_R))} \leq C_R|u|_{H_{\mathcal{O}_R}}, \quad u \in H. \quad (\tilde{\mathbf{G}}_R^*)$$

for some $C_R > 0$.

Here $V'(\mathcal{O}_R)$ is the dual space to $V(\mathcal{O}_R)$, where

$$V(\mathcal{O}_R) := \text{the closure of } \mathcal{V}(\mathcal{O}_R) \text{ in } H^1(\mathcal{O}_R, \mathbb{R}^d), \quad (6.6)$$

and $\mathcal{V}(\mathcal{O}_R)$ denotes the space of all divergence-free vector fields of class \mathcal{C}^∞ with compact supports contained in \mathcal{O}_R .

Let us recall that $H_{\mathcal{O}_R}$ is the space of restrictions to the subset \mathcal{O}_R of elements of the space H , i.e.

$$H_{\mathcal{O}_R} := \{u|_{\mathcal{O}_R}; \ u \in H\}$$

with the scalar product defined by $(u, v)_{H_{\mathcal{O}_R}} := \int_{\mathcal{O}_R} uv \, dx$, $u, v \in H_{\mathcal{O}_R}$.

Remark. From condition $(\tilde{\mathbf{G}}_R^*)$ it follows that the mapping G satisfies condition (G^{**}) in assumption (A.3).

Indeed, by estimate $(\tilde{\mathbf{G}}_R^*)$ and the continuity of the embedding $\mathcal{T}_2(Y, V'(\mathcal{O}_R)) \hookrightarrow \mathcal{L}(Y, V'(\mathcal{O}_R))$, we obtain

$$|G(u)y|_{V'(\mathcal{O}_R)} \leq C(R)|u|_{H_{\mathcal{O}_R}}\|y\|_Y, \quad u \in H, \quad y \in Y$$

for some constant $C(R) > 0$. Thus for any $\psi \in V(\mathcal{O}_R)$

$$|(G(u)y)\psi| \leq C(R)|u|_{H_{\mathcal{O}_R}}\|y\|_Y\|\psi\|_{V(\mathcal{O}_R)}, \quad u \in H, \quad y \in Y.$$

Since by definition $(\psi^{**}G(u))y := (G(u)y)\psi$, thus from the above inequality we infer that

$$|\psi^{**}G(u)|_{Y'} \leq C(R)\|\psi\|_V|u|_{H_{\mathcal{O}_R}}. \quad (6.7)$$

Therefore if we fix $\psi \in \mathcal{V}$ then there exists $R_0 > 0$ such that $\text{supp } \psi$ is a compact subset of \mathcal{O}_{R_0} . Since G is linear, estimate (6.7) with $R := R_0$ yields that the mapping

$$L_{loc}^2(\mathcal{O}, \mathbb{R}^d) \supset H \ni u \mapsto \psi^{**}G(u) \in Y'$$

is continuous in the Fréchet topology inherited on the space H from the space $L_{loc}^2(\mathcal{O}, \mathbb{R}^d)$. Thus the mapping $\psi^{**}G$ satisfies condition (G^{**}) . \square

Proof of ($\tilde{\mathbf{G}}$). Let us consider a standard orthonormal basis $h^{(i)}$, $i \in \mathbb{N}$, in $l^2(\mathbb{N})$. Let $u \in V$. Then for each $i \in \mathbb{N}$ we have

$$\begin{aligned} |G(u)h^{(i)}|_H^2 &= \left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j} + c^{(i)}u, \sum_{k=1}^d b_k^{(i)} \frac{\partial u}{\partial x_k} + c^{(i)}u \right)_H \\ &= \left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j}, \sum_{k=1}^d b_k^{(i)} \frac{\partial u}{\partial x_k} \right)_H + 2 \left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j}, c^{(i)}u \right)_H + |c^{(i)}u|_H^2. \end{aligned}$$

Thus

$$\begin{aligned} \|G(u)\|_{\mathcal{T}_2(Y,H)}^2 &= \sum_{i=1}^{\infty} |G(u)h^{(i)}|_H^2 \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j}, \sum_{k=1}^d b_k^{(i)} \frac{\partial u}{\partial x_k} \right)_H + 2 \sum_{i=1}^{\infty} \left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j}, c^{(i)}u \right)_H \\ &\quad + \sum_{i=1}^{\infty} |c^{(i)}u|_H^2. \end{aligned} \tag{6.8}$$

We estimate each term on the right-hand side of the above equality. By assumption (6.4)

$$\begin{aligned} \sum_{i=1}^{\infty} \left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j}, \sum_{k=1}^d b_k^{(i)} \frac{\partial u}{\partial x_k} \right)_H &= \int_{\mathcal{O}} \sum_{i=1}^{\infty} \sum_{j,k=1}^d b_j^{(i)}(x) b_k^{(i)}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} dx \\ &\leq (2-a) \|\nabla u\|_H^2. \end{aligned}$$

Let us move to the second term in (6.8). For each $i \in \mathbb{N}$, we have

$$\left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j}, c^{(i)}u \right)_H \leq \|b^{(i)}\|_{L^\infty} \|c^{(i)}\|_{L^\infty} \|u\| \cdot |u|_H \leq \frac{1}{2} (\|b^{(i)}\|_{L^\infty}^2 + \|c^{(i)}\|_{L^\infty}^2) \|u\| \cdot |u|_H.$$

Thus for any $\varepsilon > 0$

$$\begin{aligned} 2 \sum_{i=1}^{\infty} \left(\sum_{j=1}^d b_j^{(i)} \frac{\partial u}{\partial x_j}, c^{(i)}u \right)_H &\leq \sum_{i=1}^{\infty} (\|b^{(i)}\|_{L^\infty}^2 + \|c^{(i)}\|_{L^\infty}^2) \|u\| \cdot |u|_H \\ &= C_1 \|u\| \cdot |u|_H \leq \varepsilon \|u\|^2 + \frac{C_1^2}{4\varepsilon} |u|_H^2, \end{aligned}$$

where C_1 is defined by (6.2). The third term in (6.8) we estimate as follows

$$\sum_{i=1}^{\infty} |c^{(i)}u|_H^2 \leq \sum_{i=1}^{\infty} \|c^{(i)}\|_{L^\infty}^2 |u|_H^2 \leq C_1 |u|_H^2.$$

Hence

$$\|G(u)\|_{\mathcal{T}_2(Y,H)}^2 \leq (2 + \varepsilon - a) \|u\|^2 + \left(\frac{C_1^2}{4\varepsilon} + C_1 \right) |u|_H^2$$

and

$$\begin{aligned} 2\langle \mathcal{A}u|u \rangle - \|G(u)\|_{\mathcal{T}_2(Y,H)}^2 &\geq 2\|u\|^2 - (2 + \varepsilon - a)\|u\|^2 - \left(\frac{C_1^2}{4\varepsilon} + C_1\right)|u|_H^2 \\ &= (a - \varepsilon)\|u\|^2 - \left(\frac{C_1^2}{4\varepsilon} + C_2\right)|u|_H^2. \end{aligned}$$

It is sufficient to take $\varepsilon > 0$ such that $a - \varepsilon \in (0, 2]$. Then condition $(\tilde{\mathbf{G}})$ holds with $\eta := a - \varepsilon$ and $\lambda_0 := \frac{C_1^2}{4\varepsilon} + C_1$. \square

Proof of $(\tilde{\mathbf{G}}^*)$. Let $b = (b_1, \dots, b_d) : \overline{\mathcal{O}} \rightarrow \mathbb{R}^d$ and let $u, v \in \mathcal{V}$. Then

$$\sum_{j=1}^d \frac{\partial}{\partial x_j} (b_j u) = \sum_{j=1}^d \left(\frac{\partial b_j}{\partial x_j} u + b_j \frac{\partial u}{\partial x_j} \right) = (\operatorname{div} b)u + \sum_{j=1}^d b_j \frac{\partial u}{\partial x_j}. \quad (6.9)$$

Thus using the integration by parts formula, we obtain

$$\begin{aligned} \int_{\mathcal{O}} \left(\sum_{j=1}^d b_j \frac{\partial u}{\partial x_j} \right) v \, dx &= \sum_{j=1}^d \int_{\mathcal{O}} \frac{\partial}{\partial x_j} (b_j u) v \, dx - \int_{\mathcal{O}} (\operatorname{div} b) u v \, dx \\ &= - \sum_{j=1}^d \int_{\mathcal{O}} (b_j u) \frac{\partial v}{\partial x_j} \, dx - \int_{\mathcal{O}} (\operatorname{div} b) u v \, dx. \end{aligned}$$

Hence using the Hölder inequality, we obtain

$$\left| \int_{\mathcal{O}} \left(\sum_{j=1}^d b_j \frac{\partial u}{\partial x_j} \right) v \, dx \right| \leq \|b\|_{L^\infty} \|u\|_H \|v\|_V + \|\operatorname{div} b\|_{L^\infty} \|u\|_H \|v\|_V.$$

Therefore the bilinear form

$$\hat{b}(u, v) := \int_{\mathcal{O}} \sum_{j=1}^d b_j u \frac{\partial v}{\partial x_j} \, dx, \quad u, v \in \mathcal{V}$$

is continuous on $\mathcal{V} \times \mathcal{V} \subset H \times V$. Thus \hat{b} can be uniquely extended to the bilinear form (denoted by the same letter) $\hat{b} : H \times V \rightarrow \mathbb{R}$ and

$$|\hat{b}(u, v)| \leq (\|b\|_{L^\infty} + \|\operatorname{div} b\|_{L^\infty}) \|u\|_H \|v\|_V, \quad u \in H, \quad v \in V. \quad (6.10)$$

Hence if we define a linear map \hat{B} by $\hat{B}u := \hat{b}(u, \cdot)$, then $\hat{B}u \in V'$ for all $u \in H$ and the following inequality holds

$$|\hat{B}u|_{V'} \leq (\|b\|_{L^\infty} + \|\operatorname{div} b\|_{L^\infty}) \|u\|_H, \quad u \in H$$

Using more classical notation, we can rewrite the above inequality in the following form

$$|(b \cdot \nabla)u|_{V'} \leq (\|b\|_{L^\infty} + \|\operatorname{div} b\|_{L^\infty}) \cdot \|u\|_H, \quad u \in H. \quad (6.11)$$

Moreover,

$$|c^{(i)}u|_{V'} \leq \|c^{(i)}\|_{L^\infty} \|u\|_H, \quad u \in H. \quad (6.12)$$

Since by (6.5) $G(u)h^{(i)} = (b^{(i)} \cdot \nabla)u + c^{(i)}u$, then by the Schwarz inequality we get

$$|G(u)h^{(i)}|_{V'}^2 = |(b^{(i)} \cdot \nabla)u + c^{(i)}u|_{V'}^2 \leq 2(|(b^{(i)} \cdot \nabla)u|_{V'}^2 + |c^{(i)}u|_{V'}^2), \quad u \in H.$$

Thus by estimates (6.11) and (6.12) we obtain

$$\begin{aligned} \|G(u)h\|_{\mathcal{T}_2(Y, V')}^2 &= \sum_{i=1}^{\infty} |G(u)h^{(i)}|_{V'}^2 \leq 2 \sum_{i=1}^{\infty} (|(b^{(i)} \cdot \nabla)u|_{V'}^2 + |c^{(i)}u|_{V'}^2) \\ &\leq 2 \sum_{i=1}^{\infty} (2\|b^{(i)}\|_{L^\infty}^2 + 2\|\operatorname{div} b^{(i)}\|_{L^\infty}^2 + \|c^{(i)}\|_{L^\infty}^2) |u|_H^2, \quad u \in H. \end{aligned}$$

Hence $G(u) \in \mathcal{T}_2(Y, V')$ and

$$\|G(u)\|_{\mathcal{T}_2(Y, V')} \leq C \cdot |u|_H, \quad u \in H,$$

where $C = 2C_1$. Thus condition $(\tilde{\mathbf{G}}^*)$ holds. \square

Proof of $(\tilde{\mathbf{G}}_R^*)$. Let us fix $R > 0$. We will proceed similarly as in the proof of $(\tilde{\mathbf{G}}^*)$. Let $v \in \mathcal{V}(\mathcal{O}_R)$. Since the values of v on the boundary $\partial\mathcal{O}_R$ are equal to zero, thus using (6.9) and the integration by parts formula, we infer that

$$\begin{aligned} \int_{\mathcal{O}_R} \left(\sum_{j=1}^d b_j \frac{\partial u}{\partial x_j} \right) v \, dx &= \sum_{j=1}^d \int_{\mathcal{O}_R} \frac{\partial}{\partial x_j} (b_j u) v \, dx - \int_{\mathcal{O}_R} (\operatorname{div} b) u v \, dx \\ &= - \sum_{j=1}^d \int_{\mathcal{O}_R} (b_j u) \frac{\partial v}{\partial x_j} \, dx - \int_{\mathcal{O}_R} (\operatorname{div} b) u v \, dx. \end{aligned}$$

Using the Hölder inequality, we obtain

$$\begin{aligned} \left| \int_{\mathcal{O}_R} \left(\sum_{j=1}^d b_j \frac{\partial u}{\partial x_j} \right) v \, dx \right| &\leq \|b\|_{L^\infty} |u|_{H(\mathcal{O}_R)} \|v\|_{V(\mathcal{O}_R)} \\ &+ \|\operatorname{div} b\|_{L^\infty} |u|_{H(\mathcal{O}_R)} \|v\|_{V(\mathcal{O}_R)}. \end{aligned} \quad (6.13)$$

Therefore if we define a linear functional \hat{B}_R by

$$\hat{B}_R v := \int_{\mathcal{O}_R} \left(\sum_{j=1}^d b_j \frac{\partial u}{\partial x_j} \right) v \, dx, \quad v \in \mathcal{V}(\mathcal{O}_R),$$

we infer that it is bounded in the norm of the space $V(\mathcal{O}_R)$. Thus it can be uniquely extended to a linear bounded functional (denoted also by \hat{B}_R) on $V(\mathcal{O}_R)$. Moreover, by estimate (6.13) we have the following inequality

$$|\hat{B}_R|_{V'(\mathcal{O}_R)} \leq (\|b\|_{L^\infty} + \|\operatorname{div} b\|_{L^\infty}) |u|_{H(\mathcal{O}_R)}$$

or equivalently

$$\|(b \cdot \nabla)u\|_{V'(\mathcal{O}_R)} \leq (\|b\|_{L^\infty} + \|\operatorname{div} b\|_{L^\infty}) \cdot |u|_{H(\mathcal{O}_R)}. \quad (6.14)$$

Furthermore,

$$\|c^{(i)}u\|_{V'(\mathcal{O}_R)} \leq \|c^{(i)}\|_{L^\infty} |u|_{H_{\mathcal{O}_R}}. \quad (6.15)$$

Since by (6.5) $G(u)h^{(i)} = (b^{(i)} \cdot \nabla)u + c^{(i)}u$, thus by the Schwarz inequality we get

$$|G(u)h^{(i)}|_{V'(\mathcal{O}_R)}^2 = |(b^{(i)} \cdot \nabla)u + c^{(i)}u|_{V'(\mathcal{O}_R)}^2 \leq 2(|(b^{(i)} \cdot \nabla)u|_{V'(\mathcal{O}_R)}^2 + |c^{(i)}u|_{V'(\mathcal{O}_R)}^2).$$

Hence by estimates (6.14) and (6.15) we obtain

$$\begin{aligned} \|G(u)h\|_{\mathcal{T}_2(Y, V'(\mathcal{O}_R))}^2 &= \sum_{i=1}^{\infty} |G(u)h^{(i)}|_{V'(\mathcal{O}_R)}^2 \\ &\leq 2 \sum_{i=1}^{\infty} \left(|(b^{(i)} \cdot \nabla)u|_{V'(\mathcal{O}_R)}^2 + |c^{(i)}u|_{V'(\mathcal{O}_R)}^2 \right) \\ &\leq 2 \sum_{i=1}^{\infty} (2\|b^{(i)}\|_{L^\infty}^2 + 2\|\operatorname{div} b^{(i)}\|_{L^\infty}^2 + \|c^{(i)}\|_{L^\infty}^2) |u|_{H_{\mathcal{O}_R}}^2. \end{aligned}$$

Therefore $G(u) \in \mathcal{T}_2(Y, V'(\mathcal{O}_R))$ and

$$\|G(u)\|_{\mathcal{T}_2(Y, V'(\mathcal{O}_R))} \leq C_R \cdot |u|_{H_{\mathcal{O}_R}},$$

where $C_R = 2C_1$. Thus condition $(\tilde{\mathbf{G}}_R^*)$ holds. \square

7. 2D stochastic Navier-Stokes equations

In the two-dimensional case the martingale solution of the stochastic Navier-Stokes equation, given by Theorem 5.1, has stronger regularity properties. We will prove that \mathbb{P} -a.s. the trajectories are equal almost everywhere to a H -valued continuous functions defined on $[0, T]$. Similarly to Capiński [13] we will prove that the solutions are pathwise unique. Moreover, using the results due to Ondreját [32] we will show existence of strong solutions and uniqueness in law.

It is well-known that if $d = 2$ then the following inequality holds, see [35, Lemma III. 3.3],

$$\|u\|_{L^4} \leq 2^{\frac{1}{4}} |u|_H^{\frac{1}{2}} \|u\|^{\frac{1}{2}}, \quad u \in H_0^1. \quad (7.1)$$

In the following Lemma we recall basic properties of the form b , defined by (2.4), valid in the two-dimensional case.

Lemma 7.1. (Lemma III. 3.4 in [35]) *If $d=2$ then*

$$|b(u, v, w)| \leq 2^{\frac{1}{2}} |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \cdot \|v\| \cdot |w|^{\frac{1}{2}} \|w\|^{\frac{1}{2}}, \quad u, v, w \in V. \quad (7.2)$$

If $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ then $B(u) \in L^2(0, T; V')$ and

$$\|B(u)\|_{L^2(0, T; V')} \leq 2^{\frac{1}{2}} |u|_{L^\infty(0, T; H)} \|u\|_{L^2(0, T; V)}. \quad (7.3)$$

7.1. Regularity properties.

Lemma 7.2. *Let $d = 2$ and assume that conditions (A.1)-(A.3) are satisfied. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, u)$ be a martingale solution of problem (4.1) such that*

$$\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |u(t)|_H^2 + \int_0^T \|u(t)\|^2 dt \right] < \infty. \quad (7.4)$$

Then for $\hat{\mathbb{P}}$ -almost all $\omega \in \hat{\Omega}$ the trajectory $u(\cdot, \omega)$ is equal almost everywhere to a continuous H -valued function defined on $[0, T]$.

Proof. If u be a martingale solution of problem (4.1) then, in particular, $u \in \mathcal{C}([0, T], H_w) \cap L^2(0, T; V)$, $\hat{\mathbb{P}}$ -a.s. and

$$u(t) = u_0 - \int_0^t [Au(s) + B(u(s))] ds + \int_0^t f(s) ds + \int_0^t G(u(s)) d\hat{W}(s), \quad t \in [0, T]. \quad (7.5)$$

Let us consider the following "shifted" Stokes equations

$$z(t) = - \int_0^t Az(s) ds + \int_0^t G(u(s)) d\hat{W}(s), \quad (7.6)$$

where A is the operator defined by (2.12). Since A satisfies condition (2.13) and u satisfies inequality (7.4), by Theorem 1.3 in [33] we infer that equation (7.6) has a unique progressively measurable solution z such that $\hat{\mathbb{P}}$ -a.s. $z \in \mathcal{C}([0, T]; H)$ and

$$\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |z(t)|_H^2 + \int_0^T \|z(t)\|^2 dt \right] < \infty. \quad (7.7)$$

Let

$$v(t) := u(t) - z(t), \quad t \in [0, T].$$

For $\hat{\mathbb{P}}$ -almost all $\omega \in \hat{\Omega}$ the function $v = v(\cdot, \omega)$ is a weak solution of the following deterministic equation

$$\frac{dv(t)}{dt} = -Av(t) + v(t) + z(t) - B(v(t) + z(t)) + f(t). \quad (7.8)$$

Let $\omega \in \hat{\Omega}$ be such that $u(\cdot, \omega) \in L^2(0, T; V) \cap \mathcal{C}([0, T]; H_w)$ and $z(\cdot, \omega) \in L^2(0, T; V) \cap \mathcal{C}([0, T]; H)$. Let $\tilde{v} \in L^2(0, T; V) \cap \mathcal{C}([0, T]; H)$ be the unique solution of equation (7.8) with the initial condition $\tilde{v}(0) = u_0$ whose existence is ensured by Theorem D.1. By the uniqueness, we obtain for almost all $t \in [0, T]$

$$\tilde{v}(t) = u(t) - z(t)$$

Put

$$\hat{u}(t) := \tilde{v}(t) + z(t), \quad t \in [0, T].$$

Then $\hat{u} \in \mathcal{C}([0, T]; H)$ and $u(t) = \hat{u}(t)$ for almost all $t \in [0, T]$. This completes the proof of the lemma. \square

7.2. Uniqueness and strong solutions. Let us recall that by assumption (A.3) the mapping $G : V \rightarrow \mathcal{T}_2(Y, H)$ is Lipschitz continuous, i.e. for some $L > 0$ the following inequality holds

$$\|G(u_1(s)) - G(u_2(s))\|_{\mathcal{T}_2(Y, H)}^2 \leq L \|u_1(s) - u_2(s)\|, \quad s \in [0, T]. \quad (7.9)$$

In the following Lemma we will prove that in the case when $d = 2$ the solutions of problem (4.1) are pathwise unique. The proof is similar to that of Theorem 3.2 in [13] and uses the Schmalzfuss idea of application of the Itô formula for appropriate function, see [34].

Lemma 7.3. *Let $d = 2$ and assume that conditions (A.1)-(A.3) are satisfied. Moreover, assume that $L < 2$. If u_1 and u_2 are two solutions of problem (4.1) defined on the same filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ then $\hat{\mathbb{P}}$ -a.s. for all $t \in [0, T]$, $u_1(t) = u_2(t)$.*

Proof. Let

$$U := u_1 - u_2.$$

Then U satisfies the following equation

$$dU(t) = -\{AU(t) + [B(u_1(t)) - B(u_2(t))]\} dt + [G(u_1(t)) - G(u_2(t))] dW(t),$$

where $t \in [0, T]$. Let us define the stopping time

$$\tau_N := T \wedge \inf\{t \in [0, T] : |U(t)|_H^2 > N\}, \quad N \in \mathbb{N}. \quad (7.10)$$

Since $\hat{\mathbb{E}}[\sup_{t \in [0, T]} |U(t)|_H^2] < \infty$, $\hat{\mathbb{P}}$ -a.s. $\lim_{N \rightarrow \infty} \tau_N = T$. Let $r(t) := a \int_0^t \|u_2(s)\|^2 ds$, $t \in [0, T]$, where a is a positive constant. We apply the Itô formula to the function

$$F(t, x) = e^{-r(t)} |x|_H^2, \quad (t, x) \in [0, T] \times H.$$

Since $\frac{\partial F}{\partial t} = -r'(t)e^{-r(t)}|x|_H^2$ and $\frac{\partial F}{\partial x} = 2e^{-r(t)}x$, we obtain for all $t \in [0, T]$

$$\begin{aligned} & e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_H^2 \\ &= \int_0^{t \wedge \tau_N} e^{-r(s)} \{ -r'(s) |U(s)|_H^2 - 2 \langle AU(s) + B(u_1(s)) - B(u_2(s)), U(s) \rangle \} ds \\ &+ \frac{1}{2} \int_0^{t \wedge \tau_N} \text{Tr} \left[(G(u_1(s)) - G(u_2(s))) \frac{\partial^2 F}{\partial x^2} (G(u_1(s)) - G(u_2(s)))^* \right] ds \\ &+ 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \langle G(u_1(s)) - G(u_2(s)), U(s) dW(s) \rangle \\ &\leq \int_0^{t \wedge \tau_N} e^{-r(s)} \{ -r'(s) |U(s)|_H^2 - 2 \|U(s)\|^2 - 2 \langle B(u_1(s)) - B(u_2(s)), U(s) \rangle \} ds \\ &+ \int_0^{t \wedge \tau_N} e^{-r(s)} \|G(u_1(s)) - G(u_2(s))\|_{\mathcal{T}_2(Y, H)}^2 ds \\ &+ 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \langle G(u_1(s)) - G(u_2(s)), U(s) dW(s) \rangle. \end{aligned} \quad (7.11)$$

We have

$$B(u_1(s)) - B(u_2(s)) = B(u_1(s), U(s)) + B(U(s), u_2(s)), \quad s \in [0, T].$$

Thus by (2.9)

$$\langle B(u_1(s)) - B(u_2(s)), U(s) \rangle = \langle B(U(s), u_2(s)), U(s) \rangle$$

and hence

$$|\langle B(u_1(s)) - B(u_2(s)), U(s) \rangle| \leq \sqrt{2}|U(s)|_H \|U(s)\| \|u_2(s)\|, \quad s \in [0, T].$$

Therefore for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$2|\langle B(u_1(s)) - B(u_2(s)), U(s) \rangle| \leq \varepsilon \|U(s)\|^2 + C_\varepsilon \|u_2(s)\|^2 |U(s)|_H^2, \quad s \in [0, T]. \quad (7.12)$$

Putting $a := C_\varepsilon$, we obtain

$$\begin{aligned} & -r'(s)|U(s)|_H^2 + C_\varepsilon \|u_2(s)\|^2 |U(s)|_H^2 \\ & = -a \|u_2(s)\|^2 |U(s)|_H^2 + C_\varepsilon \|u_2(s)\|^2 |U(s)|_H^2 = 0, \quad s \in [0, T] \end{aligned}$$

and hence by (7.11) and (7.12)

$$\begin{aligned} & e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_H^2 \leq (-2 + \varepsilon) \int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|^2 ds \\ & + \int_0^{t \wedge \tau_N} e^{-r(s)} \|G(u_1(s)) - G(u_2(s))\|_{\mathcal{T}_2(Y, H)}^2 ds \\ & + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \langle G(u_1(s)) - G(u_2(s)), U(s) dW(s) \rangle, \quad t \in [0, T] \end{aligned} \quad (7.13)$$

By (7.13) and (7.9) we obtain

$$\begin{aligned} & e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_H^2 + (2 - \varepsilon - L) \int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|^2 ds \\ & \leq 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \langle G(u_1(s)) - G(u_2(s)), U(s) dW(s) \rangle, \quad t \in [0, T] \end{aligned} \quad (7.14)$$

Let choose $\varepsilon > 0$ such that $2 - \varepsilon - L > 0$. Then by (7.14), in particular, we have for all $t \in [0, T]$

$$\begin{aligned} & e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_H^2 \\ & \leq 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \langle G(u_1(s)) - G(u_2(s)), U(s) dW(s) \rangle. \end{aligned} \quad (7.15)$$

Let us denote

$$\mu(t) := \int_0^t e^{-r(s)} \langle G(u_1(s)) - G(u_2(s)), U(s) dW(s) \rangle, \quad t \in [0, T]$$

and

$$\mu_N(t) := \mu(t \wedge \tau_N), \quad t \in [0, T], \quad N \in \mathbb{N}.$$

Since by (7.9) and (7.10)

$$\begin{aligned} & \hat{\mathbb{E}} \left[\int_0^T \mathbb{1}_{[0, \tau_N]} |U(s)|_H^2 \|G(u_1(s)) - G(u_2(s))\|_{\mathcal{T}_2(Y, H)}^2 ds \right] \\ & \leq NL \hat{\mathbb{E}} \left[\int_0^T \|u_1(s) - u_2(s)\|^2 ds \right] < \infty, \end{aligned}$$

for each $N \in \mathbb{N}$ the process $\{\mu_N(t)\}$ is a martingale. In particular, by (7.14) and the martingale property we obtain for all $t \in [0, T]$

$$\hat{\mathbb{E}} \left[\int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|^2 ds \right] \leq 2\hat{\mathbb{E}}[\mu_N(t)] = 0. \quad (7.16)$$

By the Schwarz, the Burkholder-Davis-Gundy inequalities and (7.16), we obtain

$$\begin{aligned} & \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} |\mu_N(t)| \right] \\ & \leq C \hat{\mathbb{E}} \left[\left(\int_0^{T \wedge \tau_N} e^{-2r(s)} |U(s)|_H^2 \|G(u_1(s)) - G(u_2(s))\|_{\mathcal{T}_2(Y, H)}^2 ds \right)^{\frac{1}{2}} \right] \\ & \leq CL \hat{\mathbb{E}} \left[\left(\sup_{s \in [0, T]} e^{-r(s \wedge \tau_N)} |U(s \wedge \tau_N)|_H^2 \int_0^{T \wedge \tau_N} e^{-r(s)} \|U(s)\|^2 ds \right)^{\frac{1}{2}} \right] \\ & \leq \frac{1}{2} \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_H^2 + \tilde{C} \hat{\mathbb{E}} \left[\int_0^{T \wedge \tau_N} e^{-r(s)} \|U(s)\|^2 ds \right] \right] \\ & = \frac{1}{4} \hat{\mathbb{E}} \left[\sup_{t \in [0, T]} e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_H^2 \right]. \end{aligned} \quad (7.17)$$

By (7.15) and (7.17), we obtain

$$\hat{\mathbb{E}} \left[\sup_{t \in [0, T]} e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_H^2 \right] = 0.$$

Since $\hat{\mathbb{P}}$ -a.s. $\lim_{N \rightarrow \infty} \tau_N = T$ and $\hat{\mathbb{E}} \left[\int_0^T \|u_2(t)\| dt \right] < \infty$ we infer that $\hat{\mathbb{P}}$ -a.s. for all $t \in [0, T]$, $U(t) = 0$. The proof of the lemma is thus complete. \square

Definition 7.4. We say that problem (4.1) has a **strong solution** iff for every stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ and every cylindrical Wiener process $W(t)$ in a separable Hilbert space Y defined on this stochastic basis there exists a progressively measurable process $u : [0, T] \times \Omega \rightarrow H$ with \mathbb{P} -a.e. paths

$$u(\cdot, \omega) \in \mathcal{C}([0, T], H_w) \cap L^2(0, T; V)$$

such that for all $t \in [0, T]$ and all $v \in \mathcal{V}$:

$$\begin{aligned} (u(t), v)_H &+ \int_0^t \langle \mathcal{A}u(s), v \rangle ds + \int_0^t \langle B(u(s), u(s)), v \rangle ds \\ &= (u_0, v)_H + \int_0^t \langle f(s), v \rangle ds + \left\langle \int_0^t G(u(s)) d\hat{W}(s), v \right\rangle \end{aligned}$$

the identity holds $\hat{\mathbb{P}}$ - a.s.

Let us recall two basic concepts of uniqueness of the solution, i.e. pathwise uniqueness and uniqueness in law, see [22], [32].

Definition 7.5. We say that solutions of problem (4.1) are **pathwise unique** iff the following condition holds

if $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u^i)$, $i = 1, 2$, are such solutions of problem (4.1) that $u^i(0) = u_0$, $i = 1, 2$, then \mathbb{P} -a.s. for all $t \in [0, T]$, $u^1(t) = u^2(t)$ \mathbb{P} -a.s.

Definition 7.6. We say that solutions of problem (4.1) are **unique in law** iff the following condition holds

if $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, W^i, u^i)$, $i = 1, 2$, are such solutions of problem (4.1) that $u^i(0) = u_0$, $i = 1, 2$, then $\text{Law}_{\mathbb{P}^1}(u^1) = \text{Law}_{\mathbb{P}^2}(u^2)$.

Corollary 7.7. *Let $d = 2$ and assume that conditions (A.1)-(A.3) are satisfied. Moreover, assume that the Lipschitz constant L in (7.9) satisfies condition $L < 2$.*

- 1) *There exists a pathwise unique strong solution of problem (4.1)*
- 2) *Moreover, if $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u)$ is a strong solution of problem (4.1) then for \mathbb{P} -almost all $\omega \in \Omega$ the trajectory $u(\cdot, \omega)$ is equal almost everywhere to a continuous H -valued function defined on $[0, T]$.*
- 3) *The martingale solution of problem (4.1) is unique in law.*

Proof. Since by Theorem 5.1 there exists a martingale solution and by Lemma 7.3 it is pathwise unique, assertions 1) and 3) follow from theorems 2 and 12.1 in [32]. Assertion 2) is a direct consequence of Lemma 7.2. \square

Appendix A

Let us assume that the mapping $G : V \rightarrow \mathcal{T}_2(Y, H)$ satisfies the following inequality

$$2\langle \mathcal{A}u, u \rangle - \|G(u)\|_{\mathcal{T}_2(Y, H)}^2 \geq \eta \|u\|^2 - \lambda_0 |u|_H^2 - \rho, \quad u \in V \quad (\text{G})$$

for some constants $\eta \in (0, 2]$, $\lambda_0 \geq 0$ and $\rho \in \mathbb{R}$.

Since $\|u\| := \|\nabla u\|_{L^2}$ and $\langle \mathcal{A}u, u \rangle = ((u, u)) := (\nabla u, \nabla u)_{L^2}$,

$$2\langle \mathcal{A}u, u \rangle - \eta\|u\|^2 = (2 - \eta)\|u\|^2.$$

Hence inequality (G) can be written equivalently in the following form

$$\|G(u)\|_{\mathcal{T}_2(Y, H)}^2 \leq (2 - \eta)\|u\|^2 + \lambda_0|u|_H^2 + \rho, \quad u \in V. \quad (\text{G}')$$

The following proof of Lemma 5.3 is standard, see [21]. However, we provide all details to indicate the importance of the assumption (5.3) on p .

Proof of estimates (5.4), (5.5) and (5.6) in Lemma 5.3 under the assumption (G).

Let p satisfy condition (5.3), i.e.

$$\begin{aligned} p &\in \left[2, 2 + \frac{\eta}{2 - \eta}\right) \quad \text{if } \eta \in (0, 2) \\ p &\in [2, \infty) \quad \text{if } \eta = 2. \end{aligned}$$

We apply the Itô formula to the function $F(x) = |x|^p := |x|_H^p$, $x \in H$. Since

$$\frac{\partial F}{\partial x} = p \cdot |x|^{p-2} \cdot x, \quad \left\| \frac{\partial^2 F}{\partial x^2} \right\| \leq p(p-1) \cdot |x|^{p-2}, \quad x \in H,$$

we have the following equalities

$$\begin{aligned} d[|u_n(t)|^p] &= \left[p |u_n(t)|^{p-2} \langle u_n(t), -\mathcal{A}u_n(t) - B_n(u_n(t)) + P_n f(t) \rangle \right. \\ &\quad \left. + \frac{1}{2} \text{Tr} \left[P_n G(u_n(t)) \frac{\partial^2 F}{\partial x^2} (P_n G(u_n(t)))^* \right] \right] dt + p |u_n(t)|^{p-2} \langle u_n(t), G(u_n(t)) dW(t) \rangle \\ &= \left[-p |u_n(t)|^{p-2} \|u_n(t)\|^2 + p |u_n(t)|^{p-2} \langle u_n(t), f(t) \rangle \right. \\ &\quad \left. + \frac{1}{2} \text{Tr} \left[P_n G(u_n(t)) \frac{\partial^2 F}{\partial x^2} (P_n G(u_n(t)))^* \right] \right] dt \\ &\quad + p |u_n(t)|^{p-2} \langle u_n(t), G(u_n(t)) dW(t) \rangle, \quad t \in [0, T]. \end{aligned}$$

Thus

$$\begin{aligned} d[|u_n(t)|^p] + p |u_n(t)|^{p-2} \|u_n(t)\|^2 &\leq p |u_n(t)|^{p-2} \langle f, u_n(t) \rangle dt \\ &\quad + \frac{1}{2} p(p-1) |u_n(t)|^{p-2} \cdot \|P_n G(u_n(t))\|_{\mathcal{T}_2(Y, H)}^2 dt \\ &\quad + p |u_n(t)|^{p-2} \langle u_n(t), G(u_n(t)) dW(t) \rangle \quad t \in [0, T]. \end{aligned}$$

By (G')

$$\|P_n G(u_n(t))\|_{\mathcal{T}_2(Y, H)}^2 \leq (2 - \eta) \|u_n(t)\|^2 + \lambda_0 |u_n(t)|^2 + \rho.$$

Thus

$$\begin{aligned} d[|u_n(t)|^p] + p |u_n(t)|^{p-2} \|u_n(t)\|^2 &\leq p |u_n(t)|^{p-2} \langle f(t), u_n(t) \rangle dt \\ &\quad + \frac{1}{2} p(p-1) |u_n(t)|^{p-2} \cdot [(2 - \eta) \|u_n(t)\|^2 + \lambda_0 |u_n(t)|^2 + \rho] dt \\ &\quad + p |u_n(t)|^{p-2} \langle u_n(t), G(u_n(t)) dW(t) \rangle. \end{aligned}$$

Moreover, by (2.3) and the Schwarz inequality, we obtain

$$\begin{aligned} \langle f(t), u_n(t) \rangle &\leq |f(t)|_{V'} \cdot \|u_n(t)\|_V = |f(t)|_{V'} \cdot (|u_n(t)|_H^2 + \|u_n(t)\|^2)^{\frac{1}{2}} \\ &\leq |f(t)|_{V'} |u_n(t)|_H + \frac{1}{4\varepsilon} |f(t)|_{V'}^2 + \varepsilon \|u_n(t)\|^2. \end{aligned}$$

Hence

$$\begin{aligned} &d[|u_n(t)|^p] + [p - p\varepsilon - \frac{1}{2}p(p-1)(2-\eta)] |u_n(t)|^{p-2} \|u_n(t)\|^2 \\ &\leq p |u_n(t)|^{p-2} \left(|f(t)|_{V'} |u_n(t)|_H + \frac{1}{4\varepsilon} |f(t)|_{V'}^2 \right) dt \\ &\quad + \frac{1}{2}p(p-1) |u_n(t)|^{p-2} \cdot [\lambda_0 |u_n(t)|^2 + \rho] dt \\ &\quad + p |u_n(t)|^{p-2} \langle u_n(t), G(u_n(t)) dW(t) \rangle. \end{aligned}$$

Moreover,

$$\begin{aligned} &p |u_n(t)|^{p-2} \cdot \left(|f(t)|_{V'} |u_n(t)|_H + \frac{1}{4\varepsilon} |f(t)|_{V'}^2 \right) \\ &\quad + \frac{1}{2}p(p-1) |u_n(t)|^{p-2} \cdot [\lambda_0 |u_n(t)|^2 + \rho] \\ &= |u_n(t)|^{p-2} \cdot \left[\frac{1}{2}p(p-1)\lambda_0 |u_n(t)|^2 + p |f(t)|_{V'} |u_n(t)|_H + \frac{p}{4\varepsilon} |f(t)|_{V'}^2 + \frac{1}{2}p(p-1)\rho \right] \\ &\leq \left[\left(\frac{1}{2}p(p-1)\lambda_0 + \varepsilon |f(t)|_{V'}^2 \right) |u_n(t)|^p + C(\varepsilon, p, \rho) (|f(t)|_{V'}^2 + 1) \right]. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &|u_n(t)|^p + [p - p\varepsilon - \frac{1}{2}p(p-1)(2-\eta)] \int_0^t |u_n(s)|^{p-2} \|u_n(s)\|^2 ds \\ &\leq |u_n(0)|^p + \int_0^t \left(\frac{1}{2}p(p-1)\lambda_0 + \varepsilon |f(s)|_{V'}^2 \right) |u_n(s)|^p ds + C(\varepsilon, p, \rho) \int_0^t (|f(s)|_{V'}^2 + 1) ds \\ &\quad + p \int_0^t |u_n(s)|^{p-2} \langle u_n(s), G(u_n(s)) dW(s) \rangle, \quad t \in [0, T]. \end{aligned} \tag{A.1}$$

Let us choose $\varepsilon > 0$ such that $p - p\varepsilon - \frac{1}{2}p(p-1)(2-\eta) > 0$, or equivalently,

$$\varepsilon < 1 - \frac{1}{2}(p-1)(2-\eta).$$

Notice that under condition (5.3) such ε exists.

By Lemma 5.2, we infer that the process

$$\mu_n(t) := \int_0^t |u_n(s)|^{p-2} \langle u_n(s), G(u_n(s)) dW(s) \rangle, \quad t \in [0, T]$$

is a martingale and that $\mathbb{E}[\mu_n(t)] = 0$. Thus

$$\begin{aligned} \mathbb{E}[|u_n(t)|^p] &\leq \mathbb{E}[|u_n(0)|^p] + \int_0^t \left(\frac{1}{2}p(p-1)\lambda_0 + \varepsilon|f|_{V'}^2 \right) \mathbb{E}[|u_n(s)|^p] ds \\ &\quad + C(\varepsilon, p, \rho) \int_0^t (|f(s)|_{V'}^2 + 1) ds, \quad t \in [0, T] \end{aligned} \quad (\text{A.2})$$

Hence by the Gronwall Lemma there exists a constant $C > 0$ such that

$$\mathbb{E}[|u_n(t)|^p] \leq C, \quad \forall t \in [0, T] \quad \forall n \geq 1 \quad (\text{A.3})$$

Using this bound in (A.1) we also obtain

$$\mathbb{E} \left[\int_0^T |u_n(s)|^{p-2} \|u_n(s)\|^2 ds \right] \leq C, \quad n \geq 1 \quad (\text{A.4})$$

for a new constant $C > 0$. This completes the proof of estimates (5.5) and (5.6).

By the Burkholder-Davis-Gundy inequality, see [17], inequality (G') and estimates (A.3) and (A.4), we have the following inequalities

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \int_0^s p |u_n(\sigma)|^{p-2} \langle u_n(\sigma), P_n G(u_n(\sigma)) dW(\sigma) \rangle \right| \right] \\ &\leq C p \cdot \mathbb{E} \left[\left(\int_0^t |u_n(\sigma)|^{2p-2} \cdot \|G(u_n(\sigma))\|_{\mathcal{H}_2(Y,H)}^2 d\sigma \right)^{\frac{1}{2}} \right] \\ &\leq C p \cdot \mathbb{E} \left[\sup_{0 \leq \sigma \leq t} |u_n(\sigma)|^{\frac{p}{2}} \left(\int_0^t |u_n(\sigma)|^{p-2} \cdot \|G(u_n(\sigma))\|_{\mathcal{H}_2(Y,H)}^2 d\sigma \right)^{\frac{1}{2}} \right] \\ &\leq C p \cdot \mathbb{E} \left[\sup_{0 \leq \sigma \leq t} |u_n(\sigma)|^{\frac{p}{2}} \left(\int_0^t |u_n(\sigma)|^{p-2} \cdot [\lambda_0 |u_n(\sigma)|^2 + \rho + (2-\eta)\|u_n(\sigma)\|^2] d\sigma \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{2} \mathbb{E} \left[\sup_{0 \leq s \leq t} |u_n(s)|^p \right] + \frac{1}{2} C^2 p^2 \lambda_0 \cdot \mathbb{E} \left[\int_0^t \sup_{0 \leq s \leq \sigma} |u_n(s)|^p d\sigma \right] + \text{const.} \end{aligned}$$

Thus, by (A.2) we have

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq s \leq t} |u_n(s)|^p \right] &\leq \mathbb{E}[|u_n(0)|^p] + \int_0^t \left(\frac{1}{2}p(p-1)\lambda_0 + \varepsilon|f|_{V'}^2 \right) \mathbb{E} \left[\sup_{0 \leq r \leq s} |u_n(r)|^p \right] ds \\ &\quad + C(\varepsilon, p, \rho) \int_0^t (|f(s)|_{V'}^2 + 1) ds \\ &\quad + \frac{1}{2} \mathbb{E} \left[\sup_{0 \leq s \leq t} |u_n(s)|^p \right] + \frac{1}{2} C^2 p^2 \lambda_0 \cdot \int_0^t \mathbb{E} \left[\sup_{0 \leq s \leq \sigma} |u_n(s)|^p \right] d\sigma + \text{const.} \end{aligned}$$

By the Gronwall Lemma, we obtain (5.4), i.e.

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |u_n(s)|^p \right] \leq C_1(p)$$

for some constant $C_1(p) > 0$. This completes the proof of estimates (5.4), (5.5) and (5.6) in Lemma 5.3. \square

Appendix B

The following lemma is a generalization of the result contained in Corollary 5.3 in [9] for 2D domains bounded in some direction¹. Here, we consider any 2D or 3D domain, possibly unbounded, with smooth boundary. Moreover, we impose weaker assumptions on the sequence $(u_n)_n$.

Lemma B.1 *Let $u \in L^2(0, T; H)$ and let $(u_n)_n$ be a bounded sequence in $L^2(0, T; H)$ such that $u_n \rightarrow u$ in $L^2(0, T; H_{loc})$. Let $\gamma > \frac{d}{2} + 1$. Then for all $t \in [0, T]$ and all $\psi \in V_\gamma$:*

$$\lim_{n \rightarrow \infty} \int_0^t \langle B(u_n(s), u_n(s)), \psi \rangle ds = \int_0^t \langle B(u(s), u(s)), \psi \rangle ds. \quad (\text{B.1})$$

Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V_γ and V'_γ .

Proof. Assume first that $\psi \in \mathcal{V}$. Then there exists $R > 0$ such that $\text{supp } \psi$ is a compact subset of \mathcal{O}_R . Then, using the integration by parts formula, we infer that for every $v, w \in H$

$$|\langle B(v, w), \psi \rangle| = \left| \int_{\mathcal{O}_R} (v \cdot \nabla \psi) w dx \right| \leq C |u|_{H_{\mathcal{O}_R}} |w|_{H_{\mathcal{O}_R}} \|\psi\|_{V_\gamma}. \quad (\text{B.2})$$

We have $B(u_n, u_n) - B(u, u) = B(u_n - u, u_n) + B(u, u_n - u)$. Thus, using the estimate (B.2) and the Hölder inequality, we obtain

$$\begin{aligned} & \left| \int_0^t \langle B(u_n(s), u_n(s)), \psi \rangle ds - \int_0^t \langle B(u(s), u(s)), \psi \rangle ds \right| \\ & \leq \left| \int_0^t \langle B(u_n(s) - u(s), u_n(s)), \psi \rangle ds \right| + \left| \int_0^t \langle B(u(s), u_n(s) - u(s)), \psi \rangle ds \right| \\ & \leq \left(\int_0^t |u_n(s) - u(s)|_{H_{\mathcal{O}_R}} |u_n(s)|_{H_{\mathcal{O}_R}} ds \right. \\ & \quad \left. + \int_0^t |u(s)|_{H_{\mathcal{O}_R}} |u_n(s) - u(s)|_{H_{\mathcal{O}_R}} ds \right) \cdot \|\psi\|_{V_\gamma} \\ & \leq C \|u_n - u\|_{L^2(0, T; H_{\mathcal{O}_R})} (\|u_n\|_{L^2(0, T; H_{\mathcal{O}_R})} + \|u\|_{L^2(0, T; H_{\mathcal{O}_R})}) \|\psi\|_{V_\gamma}, \end{aligned}$$

where C stands for some positive constant. Since $u_n \rightarrow u$ in $L^2(0, T; H_{loc})$, we infer that (B.1) holds for every $\psi \in \mathcal{V}$.

¹A domain $\mathcal{O} \subset \mathbb{R}^d$ is bounded in some direction if there exists a nonzero vector $h \in \mathbb{R}^d$ such that $\mathcal{O} \cap (h + \mathcal{O}) = \emptyset$. Here $h + \mathcal{O} := \{h + x, x \in \mathcal{O}\}$.

If $\psi \in V_\gamma$ then for every $\varepsilon > 0$ there exists $\psi_\varepsilon \in \mathcal{V}$ such that $\|\psi - \psi_\varepsilon\|_{V_\gamma} \leq \varepsilon$. Then

$$\begin{aligned}
& |\langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi \rangle| \\
& \leq |\langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi - \psi_\varepsilon \rangle| \\
& \quad + |\langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi_\varepsilon \rangle| \\
& \leq (|B(u_n(\sigma), u_n(\sigma))|_{V_\gamma'} + |B(u(\sigma), u(\sigma))|_{V_\gamma'}) \|\psi - \psi_\varepsilon\|_{V_\gamma} \\
& \quad + |\langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi_\varepsilon \rangle| \\
& \leq \varepsilon (|u_n(\sigma)|_H^2 + |u(\sigma)|_H^2) + |\langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi_\varepsilon \rangle|.
\end{aligned}$$

Hence

$$\begin{aligned}
& \left| \int_s^t \langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi \rangle d\sigma \right| \\
& \leq \varepsilon \int_0^t (|u_n(\sigma)|_H^2 + |u(\sigma)|_H^2) d\sigma \\
& \quad + \left| \int_s^t \langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi_\varepsilon \rangle d\sigma \right| \\
& \leq \varepsilon \cdot \left(\sup_{n \geq 1} \|u_n\|_{L^2(0,T;H)}^2 + \|u\|_{L^2(0,T;H)}^2 \right) \\
& \quad + \left| \int_s^t \langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi_\varepsilon \rangle d\sigma \right|.
\end{aligned}$$

Passing to the upper limit as $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} \left| \int_s^t \langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi \rangle d\sigma \right| \leq M\varepsilon,$$

where $M := \sup_{n \geq 1} \|u_n\|_{L^2(0,T;H)}^2 + \|u\|_{L^2(0,T;H)}^2 < \infty$. Since $\varepsilon > 0$ is arbitrary, we infer that

$$\lim_{n \rightarrow \infty} \int_s^t \langle B(u_n(\sigma), u_n(\sigma)) - B(u(\sigma), u(\sigma)), \psi \rangle d\sigma = 0.$$

This completes the proof. \square

In the following Corollary we state a result which is used in the proof of Lemma 5.5.

Corollary B.2 *Let $u \in L^2(0, T; H)$ and let $(u_n)_n$ be a bounded sequence in $L^2(0, T; H)$ such that $u_n \rightarrow u$ in $L^2(0, T; H_{loc})$. Then for all $t \in [0, T]$ and all $\psi \in U$:*

$$\lim_{n \rightarrow \infty} \int_0^t \langle B(u_n(s), u_n(s)), P_n \psi \rangle ds = \int_0^t \langle B(u(s), u(s)), \psi \rangle ds. \quad (\text{B.3})$$

Proof. Let $t \in [0, T]$ and let $\psi \in U$. We have

$$\begin{aligned} & \int_0^t \langle B(u_n(s), u_n(s)), P_n \psi \rangle ds \\ &= \int_0^t \langle B(u_n(s), u_n(s)), P_n \psi - \psi \rangle ds + \int_0^t \langle B(u_n(s), u_n(s)), \psi \rangle ds \\ &=: I_1(n) + I_2(n). \end{aligned}$$

Let $\gamma > \frac{d}{2} + 1$. Let us consider first the term $I_1(n)$. By (2.10), we have the following inequalities

$$\begin{aligned} |I_1(n)| &\leq \int_0^t |B(u_n(s), u_n(s))|_{V_\gamma} ds \cdot \|P_n \psi - \psi\|_{V_\gamma} \\ &\leq c \int_0^T |u_n(s)|_H^2 ds \cdot \|P_n \psi - \psi\|_{V_\gamma}. \end{aligned}$$

Since the sequence $(u_n)_{n \geq 1}$ is bounded in $L^2(0, T; H)$, by (ii) in Lemma 2.4 (c) we infer that

$$\lim_{n \rightarrow \infty} I_1(n) = 0.$$

Since $U \subset V_\gamma$, by Lemma B.1 we infer that

$$\lim_{n \rightarrow \infty} I_2(n) = \int_0^t \langle B(u(s), u(s)), \psi \rangle ds.$$

The proof of the Lemma is thus complete. \square

Appendix C: Some auxilliary results from functional analysis

The following result can be found in Holly and Wiciak, [20].

Lemma C.1 (see Lemma 2.5, p.99 in [20]) *Consider a separable Banach space Φ having the following property*

$$\text{there exists a Hilbert space } H \text{ such that } \Phi \subset H \text{ continuously.} \quad (\text{C.1})$$

Then there exists a Hilbert space $(\mathcal{H}, (\cdot | \cdot)_{\mathcal{H}})$ such that $\mathcal{H} \subset \Phi$, \mathcal{H} is dense in Φ and the embedding $\mathcal{H} \hookrightarrow \Phi$ is compact.

Proof. Without loss of generality we can assume that $\dim \Phi = \infty$ and Φ is dense in H . Since Φ is separable, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \Phi$ linearly dense in Φ . Since Φ is dense in H and the embedding $\Phi \hookrightarrow H$ is continuous, the subspace $\text{span}\{\varphi_1, \varphi_2, \dots\}$ is dense in H . After the orthonormalisation of (φ_n) in the Hilbert space $(H, (\cdot | \cdot)_H)$ we obtain an orthonormal basis (h_n) of this space. Furthermore, the sequence (h_n) is linearly dense in Φ . Since the natural embedding $\iota : \Phi \hookrightarrow H$ is continuous, we infer that

$$1 = |h_n|_H = |\iota(h_n)|_H \leq |\iota| \cdot |h_n|_\Phi$$

and $\frac{1}{|h_n|_\Phi} \leq |\iota|$ for all $n \in \mathbb{N}$.

Let us take $\eta_0 \in (0, 1)$ and define inductively a sequence $(\eta_n)_{n \in \mathbb{N}}$ by

$$\eta_n := \frac{\eta_{n-1} + 1}{2}, \quad n = 1, 2, \dots$$

The sequence (η_n) is strongly increasing and $\lim_{n \rightarrow \infty} \eta_n = 1$. Let us define a sequence $(r_n)_{n \in \mathbb{N}}$ by

$$r_n := \frac{1 - \eta_n}{2|h_n|_\Phi} > 0, \quad n = 1, 2, \dots$$

Obviously $\lim_{n \rightarrow \infty} r_n = 0$. Let us consider the set

$$\mathcal{H} := \left\{ x \in H : \sum_{n=1}^{\infty} \frac{1}{r_n^2} \cdot |(x|h_n)_H|^2 < \infty \right\}$$

and the Hilbert space $L_\mu^2(\mathbb{N}^*, \mathbb{K})$, where $\mu : 2^{\mathbb{N}^*} \rightarrow [0, \infty]$ is the measure given by the formula

$$\mu(M) := \sum_{n \in M} \frac{1}{r_n^2}, \quad M \subset \mathbb{N}^*.$$

The linear operator

$$l : L_\mu^2(\mathbb{N}^*, \mathbb{K}) \ni \xi \mapsto \sum_{n=1}^{\infty} \xi_n h_n \in H$$

is well defined. Moreover, l is an injection and hence we may introduce the following inner product

$$(\cdot | \cdot)_{\mathcal{H}} := (\cdot | \cdot)_{L^2} \circ l^{-1} : \mathcal{H} \times \mathcal{H} \ni (x, y) \mapsto (l^{-1}x | l^{-1}y)_{L^2} \in \mathbb{K}.$$

Now, l is an isometry onto the pre-Hilbert space $(\mathcal{H}, (\cdot | \cdot)_{\mathcal{H}})$ and consequently \mathcal{H} is $(\cdot | \cdot)_{\mathcal{H}}$ -complete. Let us notice that for all $x, y \in \mathcal{H}$

$$(x|y)_{\mathcal{H}} = \sum_{n=1}^{\infty} \frac{1}{r_n^2} \cdot (x|h_n)_H (y|h_n)_H, \quad |x|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \frac{1}{r_n^2} \cdot |(x|h_n)_H|^2$$

We will show that $\mathcal{H} \subset \Phi$ continuously. Indeed, let $x \in \mathcal{H}$, $|x|_{\mathcal{H}} \leq 1$. Then for $i \in \mathbb{N}$

$$|(x|h_i)h_i|_\Phi = |(x|h_i)| \cdot |h_i|_\Phi \leq r_i |h_i|_\Phi = \frac{1 - \eta_{i-1}}{2|h_i|_\Phi} |h_i|_\Phi = \frac{1 - \eta_{i-1}}{2} = \eta_i - \eta_{i-1}.$$

Thus, for any $k, n \in \mathbb{N}$, $k < n$, we have the following estimate

$$\left| \sum_{i=k+1}^n (x|h_i)h_i \right|_\Phi \leq \sum_{i=k+1}^n (\eta_i - \eta_{i-1}) = \eta_n - \eta_k.$$

Since in particular, the sequence $(s_n := \sum_{i=1}^n (x|h_i)h_i)$ is Cauchy in the Banach space $(\Phi, |\cdot|_\Phi)$, there exists $\varphi \in \Phi$ such that $\lim_{n \rightarrow \infty} |s_n - \varphi|_\Phi = 0$.

On the other hand, $s_n = \sum_{i=1}^n (x|h_i)h_i \rightarrow x$ in H . Thus by the uniqueness of the limit $\varphi = x \in \Phi$ and

$$\sum_{i=1}^n (x|h_i)h_i \rightarrow x \quad \text{in } \Phi.$$

Moreover,

$$|x|_\Phi \xleftarrow{\infty \leftarrow n} |s_n|_\Phi \leq \eta_n - \eta_0 \xrightarrow{n \rightarrow \infty} 1 - \eta_0.$$

Thus $\mathcal{H} \subset \Phi$ continuously (with the norm of the embedding not exceeding $1 - \eta_0$).

We will show that the embedding $j : \mathcal{H} \hookrightarrow \Phi$ is compact. It is sufficient to prove that the ball $Z := \{x \in \mathcal{H} : |x|_\mathcal{H} \leq 1\}$ is relatively compact in $(\Phi, |\cdot|_\Phi)$. According to the Hausdorff Theorem it is sufficient to find (for every fixed ε) an ε -net of the set $j(Z)$.

Since $\lim_{n \rightarrow \infty} \eta_n = 1$, there exists $n \in \mathbb{N}$ such that $1 - \eta_n \leq \frac{\varepsilon}{2}$. The linear operator

$$S_n : \mathcal{H} \ni x \mapsto \sum_{i=1}^n (x|h_i)h_i \in \Phi$$

being finite-dimensional is compact. Therefore $S_n(Z)$ is relatively compact in $(\Phi, |\cdot|_\Phi)$ and consequently there is a finite subset $F \subset \Phi$ such that $S_n(Z) \subset \bigcup_{\varphi \in F} \mathbb{B}_\Phi(\varphi, \frac{\varepsilon}{2})$.

We will show that the set F is the ε -net for $j(Z)$. Indeed, let $x \in Z$. Then $S_N(x) \rightarrow x$ in $(\Phi, |\cdot|_\Phi)$ and

$$|x - S_N(x)|_\Phi \xleftarrow{\infty \leftarrow N} |S_N(x) - S_n(x)|_\Phi \leq \eta_N - \eta_n \xrightarrow{N \rightarrow \infty} 1 - \eta_n \leq \frac{\varepsilon}{2}.$$

On the other hand, $S_n(x) \in S_n(Z)$, so, there is $\varphi \in F$ such that $S_n(x) \in \mathbb{B}_\Phi(\varphi, \frac{\varepsilon}{2})$. Finally,

$$|x - \varphi|_\Phi \leq |x - S_n(x)|_\Phi + |S_n(x) - \varphi|_\Phi \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e. $x \in \mathbb{B}_\Phi(\varphi, \varepsilon)$. Thus

$$Z \subset \bigcup_{\varphi \in F} \mathbb{B}_\Phi(\varphi, \varepsilon).$$

The proof is thus complete. \square

Appendix D: Auxiliary deterministic result

Assume that $d = 2$. Let $z \in L^2(0, T, V) \cap \mathcal{C}([0, T]; H)$, $f \in L^2(0, T; V')$ and $u_0 \in H$ be given. Let us consider the following problem

$$\begin{cases} \frac{dv(t)}{dt} = -Av(t) + v(t) + z(t) - B(v(t) + z(t)) + f(t), & t \in (0, T), \\ v(0) = u_0. \end{cases} \quad (\text{D.1})$$

Definition D.1 We say that $v \in L^2(0, T, V) \cap \mathcal{C}([0, T]; H_w)$ is a **weak solution** of problem (D.1) iff for all $t \in [0, T]$ and all $\phi \in V$ the following equality holds

$$\begin{aligned} (v(t), \phi)_H &= (u_0, \phi)_H - \int_0^t (v(s), \phi)_V ds + \int_0^t (v(s) + z(s), \phi)_H ds \\ &\quad - \int_0^t \langle B(v(s) + z(s)), \phi \rangle ds + \int_0^t \langle f(s), \phi \rangle ds. \end{aligned} \quad (\text{D.2})$$

Using the Galerkin method and the compactness criterion contained in Lemma 3.3 we will prove the existence and uniqueness of the weak solutions. Moreover, using Lemma III.1.2 in [35] we show that the weak solution is almost everywhere equal to a continuous H -valued function.

Theorem D.1 Assume that $d = 2$. Let $z \in L^2(0, T, V) \cap \mathcal{C}([0, T]; H)$, $f \in L^2(0, T; V')$ and $u_0 \in H$.

- (a) There exists a unique weak solution of problem (D.1).
- (b) The weak solution is almost everywhere equal to a continuous H -valued function defined on $[0, T]$.

Proof. Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis in H composed of eigenvectors of the operator L defined by (2.19). Let $H_n := \text{span}\{e_1, \dots, e_n\}$ be the subspace with the norm inherited from H and let $P_n : U' \rightarrow H_n$ be defined by (2.25). Let us consider the Faedo-Galerkin approximation in the space H_n

$$\begin{cases} \frac{dv_n(t)}{dt} = P_n[-Av_n(t) + v_n(t) + z(t) - B(v_n(t) + z(t)) + f(t)], \\ v_n(0) = P_n u_0. \end{cases} \quad (\text{D.3})$$

In a standard way we infer that there exists a unique absolutely continuous local solution of problem (D.3). Moreover, by Lemma 2.2 (a), we obtain

$$\frac{d}{dt} |v_n(t)|_H^2 = -2\|v_n(t)\|^2 + 2(z(t), v_n(t))_H - 2\langle B(v_n(t) + z(t)), v_n(t) \rangle + 2\langle f(t), v_n(t) \rangle. \quad (\text{D.4})$$

Step 1. We establish some estimates on the Galerkin solutions v_n . Let us fix $n \in \mathbb{N}$ and $t \in (0, T)$. We have the following inequalities

$$2(z(t), v_n(t))_H \leq 2|z(t)|_H |v_n(t)|_H \leq |z(t)|_H^2 + |v_n(t)|_H^2, \quad (\text{D.5})$$

and

$$2\langle f(t), v_n(t) \rangle \leq 2|f(t)|_{V'} \|v_n(t)\|_V \leq |f(t)|_{V'}^2 + |v_n(t)|_H^2 + \|v_n(t)\|^2. \quad (\text{D.6})$$

By (2.9) and (2.8), we obtain

$$\begin{aligned} \langle B(v_n(t) + z(t)), v_n(t) \rangle &= \langle B(v_n(t) + z(t)), z(t) \rangle, v_n(t) \rangle \\ &= -\langle B(v_n(t) + z(t), v_n(t)), z(t) \rangle. \end{aligned}$$

We claim that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|\langle B(v_n(t) + z(t), v_n(t)), z(t) \rangle| \leq \varepsilon \|v_n(t)\|^2 + C_\varepsilon (\|v_n(t)\|_H^2 + 1) \|z(t)\|_{L^4}^4. \quad (\text{D.7})$$

Indeed, we have

$$\begin{aligned} & |\langle B(v_n(t) + z(t), v_n(t)), z(t) \rangle| \\ & \leq \|v_n(t) + z(t)\|_{L^4} \|z(t)\|_{L^4} \|v_n(t)\| \\ & \leq \frac{\varepsilon}{2} \|v_n(t)\|^2 + \tilde{C}_\varepsilon \|v_n(t) + z(t)\|_{L^4}^2 \|z(t)\|_{L^4}^2 \end{aligned} \quad (\text{D.8})$$

for some constant $\tilde{C}_\varepsilon > 0$. Moreover, by (7.1) we obtain

$$\begin{aligned} & \tilde{C}_\varepsilon \|v_n(t) + z(t)\|_{L^4}^2 \|z(t)\|_{L^4}^2 \\ & \leq 2\tilde{C}_\varepsilon (\|v_n(t)\|_{L^4}^2 + \|z(t)\|_{L^4}^2) \|z(t)\|_{L^4}^2 \\ & \leq 2\sqrt{2}\tilde{C}_\varepsilon \|v_n(t)\|_H \|v_n(t)\| \cdot \|z(t)\|_{L^4}^2 + 2\tilde{C}_\varepsilon \|z(t)\|_{L^4}^4 \\ & \leq \frac{\varepsilon}{2} \|v_n(t)\|^2 + \tilde{C}_\varepsilon \|v_n(t)\|_H^2 \|z(t)\|_{L^4}^4 + 2\tilde{C}_\varepsilon \|z(t)\|_{L^4}^4. \end{aligned} \quad (\text{D.9})$$

By (D.8) and (D.9) the proof of inequality (D.7) is complete.

Using inequalities (D.5), (D.6) and inequality (D.7) with $\varepsilon := \frac{1}{2}$ in (D.4) we infer that

$$\frac{d}{dt} |v_n(t)|_H^2 + \frac{1}{2} \|v_n(t)\|^2 \leq a(t) + \theta(t) |v_n(t)|_H^2, \quad (\text{D.10})$$

where $a(t) = |z(t)|_H^2 + |f(t)|_V^2 + C_{\frac{1}{2}} \|z(t)\|_{L^4}^4$ and $\theta(t) = 2 + C_{\frac{1}{2}} \|z(t)\|_{L^4}^4$. Since $z \in L^2(0, T, V) \cap \mathcal{C}([0, T]; H)$, by inequality (7.1) we have $\|z(t)\|_{L^4}^4 \leq 2|z(t)|_H^2 \|z(t)\|^2$, $t \in [0, T]$, and hence

$$\int_0^T \|z(t)\|_{L^4}^4 dt \leq 2 \sup_{t \in [0, T]} |z(t)|_H^2 \int_0^T \|z(t)\|^2 dt < \infty.$$

Notice that the functions a and θ are nonnegative and integrable.

By (D.10), we obtain

$$|v_n(t)|_H^2 + \frac{1}{2} \int_0^t \|v_n(s)\|^2 ds \leq \int_0^t a(s) ds + \int_0^t \theta(s) |v_n(s)|_H^2 ds. \quad (\text{D.11})$$

In particular,

$$|v_n(t)|_H^2 \leq \int_0^t a(s) ds + \int_0^t \theta(s) |v_n(s)|_H^2 ds$$

and hence by the Gronwall Lemma, see [37, page 18], for all $t \in \text{dom}(v_n)$

$$\begin{aligned} & |v_n(t)|_H^2 \leq |v_n(0)|_H^2 \exp\left(\int_0^t \theta(s) ds\right) + \int_0^t a(s) \exp\left(\int_s^t \theta(r) dr\right) ds \\ & \leq |u_0|_H^2 \exp\left(\int_0^T \theta(s) ds\right) + \exp\left(\int_0^T \theta(r) dr\right) \int_0^T a(s) ds =: K_1. \end{aligned} \quad (\text{D.12})$$

Since inequality (D.12) holds for all $t \in \text{dom}(v_n)$, we infer that

$$\sup_{t \in \text{dom}(v_n)} |v_n(t)|_H^2 \leq K_1. \quad (\text{D.13})$$

By (D.11) and (D.13) we have also the following inequality

$$\int_{\text{dom}(v_n)} \|v_n(s)\|^2 ds \leq K_2$$

for some constant $K_2 > 0$. Hence $\text{dom}(v_n) = [0, T]$, i.e. the Galerkin solutions are defined on the whole interval $[0, T]$ and satisfy the following inequalities

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |v_n(t)|_H^2 \leq K_1. \quad (\text{D.14})$$

and

$$\sup_{n \in \mathbb{N}} \int_0^T \|v_n(s)\|^2 ds \leq K_2 \quad (\text{D.15})$$

Step 2. Let us consider the sequence (v_n) of the Galerkin solutions. Using Lemma 3.3, we will show that (v_n) is relatively compact in the space \mathcal{Z} defined by (3.14), i.e.

$$\mathcal{Z} = \mathcal{C}([0, T]; U') \cap L_w^2(0, T; V) \cap L^2(0, T; H_{loc}) \cap \mathcal{C}([0, T]; H_w).$$

Indeed, since (v_n) satisfies inequalities (D.14) and (D.15), it is sufficient to check that (v_n) satisfies condition (c) in Lemma 3.3.

By (D.3) and Lemma 2.2 (a) we have for all $s, t \in [0, T]$ such that $s \leq t$

$$\begin{aligned} v_n(t) - v_n(s) = & - \int_s^t P_n \mathcal{A} v_n(r) dr + \int_s^t P_n z(r) dr \\ & - \int_s^t P_n B(v_n(r) + z(r)) dr + \int_s^t P_n f(r) dr. \end{aligned} \quad (\text{D.16})$$

Let us recall that we have the following continuous embeddings

$$U \subset V_\gamma \subset V \subset H \cong H' \subset U',$$

where $\gamma > \frac{d}{2} + 1$. Let us fix $\delta > 0$ and assume that $|t - s| \leq \delta$. We will estimate each term on the right-hand side of equality (D.16).

By continuity of the embedding $U \subset V$, (2.14), the Hölder inequality, Lemma 2.4 (c) and (D.15) we have

$$\left| \int_s^t P_n \mathcal{A} v_n(r) dr \right|_{U'} \leq c \delta^{\frac{1}{2}} \left(\int_s^t \|v_n(r)\|^2 dr \right)^{\frac{1}{2}} \leq c_1 \delta^{\frac{1}{2}}. \quad (\text{D.17})$$

Since the embedding $U \subset H$ is continuous and the restriction of P_n to U in an orthogonal projection and $z \in \mathcal{C}([0, T]; H)$, we have

$$\left| \int_s^t P_n z(r) dr \right|_{U'} \leq c \sup_{r \in [0, T]} |z(r)|_H \cdot \delta = c_2 \delta \quad (\text{D.18})$$

By the continuity of the embedding $U \subset V_\gamma$, Lemma 2.4 (c) and inequalities (2.10) and (D.14) we obtain

$$\begin{aligned} \left| \int_s^t P_n B(v_n(r) + z(r)) dr \right|_{U'} &\leq C \int_s^t |v_n(r) + z(r)|_H^2 dr \\ &\leq C \delta \sup_{r \in [0, T]} |v_n(r) + z(r)|_H^2 \leq c_3 \delta. \end{aligned} \quad (\text{D.19})$$

Since the embedding $U \subset V$ is continuous and $f \in L^2(0, T; V')$, by the Hölder inequality and Lemma 2.4 (c) we obtain

$$\left| \int_s^t P_n f(r) dr \right|_{U'} \leq c \delta^{\frac{1}{2}} \left(\int_s^t |f(r)|_{V'}^2 dr \right)^{\frac{1}{2}} \leq c_4 \delta^{\frac{1}{2}}. \quad (\text{D.20})$$

By (D.16) and inequalities (D.17)-(D.20) we infer that

$$\lim_{\delta \rightarrow 0} \sup_{n \in \mathbb{N}} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \delta}} |v_n(t) - v_n(s)|_{U'} = 0.$$

Hence by Lemma 3.3 the sequence (v_n) contains a subsequence, still denoted by (v_n) , convergent in the space \mathcal{Z} to some $v \in \mathcal{Z}$. In particular, $v \in L^2(0, T; V) \cap \mathcal{C}([0, T]; H_w)$.

Step 3. We will prove that v satisfies equality (D.2). By (D.3), Lemma 2.2 (a) and equalities (2.15) and (2.27), we infer that for all $\phi \in U$

$$\begin{aligned} (v_n(t), \phi)_H &= (u_0, P_n \phi)_H - \int_0^t ((v_n(s), P_n \phi)) ds + \int_0^t (z(s), P_n \phi)_H ds \\ &\quad - \int_0^t \langle B(v_n(s) + z(s)), P_n \phi \rangle ds + \int_0^t \langle f(s), P_n \phi \rangle ds. \end{aligned} \quad (\text{D.21})$$

We will pass to the limit as $n \rightarrow \infty$ in equality (D.21). Since $v_n \rightarrow v$ in the space \mathcal{Z} , by Lemmas 2.4 (c) and B.2 we find that for all $t \in [0, T]$ and $\phi \in U$

$$\begin{aligned} (v(t), \phi)_H &= (u_0, \phi)_H - \int_0^t ((v(s), \phi)) ds + \int_0^t (z(s), \phi)_H ds \\ &\quad - \int_0^t \langle B(v(s) + z(s)), \phi \rangle ds + \int_0^t \langle f(s), \phi \rangle ds. \end{aligned} \quad (\text{D.22})$$

Since U is dense in the space V , equality holds for all $\phi \in V$, as well. Hence by (2.2), v satisfies equality (D.2), i.e. v is a weak solution of problem (D.1).

Step 4. (Uniqueness) Let v_1, v_2 be two weak solutions of problem (D.1). Let $w := v_1 - v_2$. Since

$$B(v_1 + z) - B(v_2 + z) = B(v_1 + z, w) + B(w, v_2 + z),$$

w satisfies the following equation

$$\begin{cases} \frac{dw}{dt} = -\mathcal{A}w - B(v_1 + z, w) - B(w, v_2 + z), \\ w(0) = 0. \end{cases}$$

By Lemma 7.1, we infer that $w' \in L^2(0, T; V')$ and hence

$$\frac{1}{2} \frac{d}{dt} |w(t)|_H^2 = -\|w(t)\|^2 - \langle B(w, v_2 + z), w \rangle.$$

Moreover, by (7.2)

$$\begin{aligned} |-\langle B(w, v_2 + z), w \rangle| &= |\langle B(w), v_2 + z \rangle| \leq \sqrt{2} |w|_H \|w\| \|v_2 + z\| \\ &\leq \frac{1}{2} \|w\|^2 + |w|_H^2 \|v_2 + z\|^2. \end{aligned}$$

Thus

$$\frac{d}{dt} |w(t)|_H^2 + \|w(t)\|^2 \leq 2 |w(t)|_H^2 \|v(t)_2 + z(t)\|^2$$

and, in particular,

$$|w(t)|_H^2 \leq |w(0)|_H^2 + 2 \int_0^t |w(s)|_H^2 \|v(s)_2 + z(s)\|^2 ds, \quad t \in [0, T].$$

Since $w(0) = 0$, by the Gronwall Lemma we infer that $w = 0$, i.e. $v_1 = v_2$.

Let us move to the proof of (b). Let us write equality (D.22) in the following form

$$\frac{dv}{dt} = -\mathcal{A}v + z - B(v + z) + f.$$

Since $v \in L^2(0, T; V)$, by Lemma III.1.2 in [35] it is sufficient to show that $v' \in L^2(0, T; V')$. The most difficulty appears in the nonlinear term. However, since $v, z \in L^2(0, T; V) \cap L^\infty(0, T; H)$, Lemma 7.1 yields that $B(v + z) \in L^2(0, T; V')$. The proof of the Theorem is thus complete. \square

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